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Rotational Surfaces with Pointwise 1-Type Gauss Map in Pseudo Euclidean Space $\mathbb{E}_2^4$

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Abstract. In this paper, we study rotational surfaces of elliptic, hyperbolic and parabolic type with pointwise 1-type Gauss map which have spacelike profile curve in four dimensional pseudo Euclidean space $\mathbb{E}_2^4$ and obtain some characterizations for these rotational surfaces to have pointwise 1-type Gauss map.
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1 Introduction

The Gauss map G of a submanifold M into $G(n,m)$ in $\wedge^n\mathbb{E}_s^m$, where $G(n,m)$ is the Grassmannian manifold consisting of all oriented $n-$planes through the origin of $\mathbb{E}_s^m$ and $\wedge^n\mathbb{E}_s^m$ is the vector space obtained by the exterior product of n vectors in $\mathbb{E}_s^m$ is a smooth map which carries a point p in M into the oriented $n-$plane in $\mathbb{E}_s^m$ obtained from parallel translation of the tangent space of M at p in $\mathbb{E}_s^m$. Since the vector space $\wedge^n\mathbb{E}_s^m$ identify with a semi-Euclidean space $\mathbb{E}_t^N$ for some positive integer t, where $N = \binom{m}{n}$, the Gauss map is defined by $G : M \to G(n,m) \subset \mathbb{E}_t^N$, $G(p) = (e_1 \wedge ... \wedge e_n)(p)$. The notion of submanifolds with finite type Gauss map was introduced by B. Y.Chen and P.Piccinni in 1987 [6] and after then many works were done about this topic, especially 1-type Gauss map and 2- type Gauss map.

If a submanifold M of a Euclidean space or pseudo-Euclidean space has 1-type Gauss map G, then G satisfies

$$\Delta G = \lambda (G + C)$$

for some $\lambda \in \mathbb{R}$ and some constant vector C.

On the other hand the Laplacian of the Gauss map of some typical well-known surfaces satisfies the form

$$\Delta G = f\,(G + C) \tag{1.1}$$

for some smooth function f on M and some constant vector C. A submanifold of a Euclidean space or pseudo-Euclidean space is said to have pointwise 1-type Gauss map, if its Gauss map satisfies (1.1) for some smooth function f on M and some constant vector C. If the vector C in (1.1) is zero, a submanifold with pointwise 1-type Gauss map is said to be of the first kind, otherwise it is said to be of the second kind.

A lot of papers were recently published about rotational surfaces with pointwise 1-type Gauss map in four dimensional Euclidean and pseudo Euclidean space in [1],[3],[4], [8], [9] [11].Timelike and spacelike rotational surfaces of elliptic, hyperbolic and parabolic types in Minkowski space $\mathbb{E}^4_1$ with pointwise 1-type Gauss map were studied in [5, 7]. Aksoyak and Yaylı in [2] studied boost invariant surfaces (rotational surfaces of hyperbolic type) with pointwise 1-type Gauss map in Minkowski space $\mathbb{E}^4_1$. They gave a characterization for flat boost invariant surfaces with pointwise 1-type Gauss map. Also they obtain some results for boost invariant marginally trapped surfaces with pointwise 1-type Gauss map. Ganchev and Milousheva in [10] defined three types of rotational surfaces with two dimensional axis rotational surfaces of elliptic, hyperbolic and parabolic type in pseudo Euclidean space $\mathbb{E}^4_2$. They classify all rotational marginally trapped surfaces of elliptic, hyperbolic and parabolic type, respectively.

In this paper, we study rotational surfaces of elliptic, hyperbolic and parabolic type with pointwise 1-type Gauss map which have spacelike profile curve in four dimensional pseudo Euclidean space and give all classifications of flat rotational surfaces of elliptic, hyperbolic and parabolic type with pointwise 1-type Gauss map.

2 Preliminaries

Let $\mathbb{E}^m_s$ be the $m-$dimensional pseudo-Euclidean space with signature $(s, m-s)$. Then the metric tensor g in $\mathbb{E}^m_s$ has the form

$$g = \sum_{i=1}^{m-s} (dx_i)^2 - \sum_{i=m-s+1}^{m} (dx_i)^2$$

where $(x_1, \ldots, x_m)$ is a standard rectangular coordinate system in $\mathbb{E}^m_s$.

A vector v is called spacelike (resp., timelike) if $\langle v, v\rangle > 0$ (resp., $\langle v, v\rangle < 0$). Avector v is called lightlike if it $v \neq 0$ and $\langle v, v\rangle = 0$, where $\langle , \rangle$ is indefinite inner scalar product with respect to g.

Let M be an $n-$dimensional pseudo-Riemannian submanifold of a $m-$dimensional pseudo-Euclidean space $\mathbb{E}_s^m$ and denote by $\tilde{\nabla}$ and ∇ Levi-Civita connections of $\mathbb{E}_s^m$ and M , respectively. We choose local orthonormal frame $\{e_1, \ldots, e_n,\ \ e_{n+1}, \ldots, e_m\}$ on M with $\varepsilon_A = \langle e_A, e_A \rangle = \pm 1$ such that $e_1, \ldots e_n$ are tangent to M and $e_{n+1}, \ldots, e_m$ are normal to M. We use the following convention on the ranges of indices: $1 \leq i, j, k, \ldots \leq n$, $n+1 \leq r, s, t, \ldots \leq m$, $1 \leq A, B, C, \ldots \leq m$.

Denote by ω_A the dual-1 form of e_A such that $\omega_A(X) = \langle e_A, X \rangle$ and ω_{AB} the connection forms defined by

$$de_A = \sum_B \varepsilon_B \omega_{AB} e_B, \quad \omega_{AB} + \omega_{BA} = 0.$$

Then the formulas of Gauss and Weingarten are given by

$$\tilde{\nabla}_{e_k} e_i = \sum_{j=1}^{n} \varepsilon_j \omega_{ij}(e_k)\, e_j + \sum_{r=n+1}^{m} \varepsilon_r h_{ik}^r e_r$$

and

$$\tilde{\nabla}_{e_k} e_s = -\sum_{j=1}^{n} \varepsilon_j h_{kj}^s e_j + D_{e_k} e_s, \quad D_{e_k} e_s = \sum_{r=n+1}^{m} \varepsilon_r \omega_{sr}(e_k)\, e_r,$$

where D is the normal connection, h_{ik}^r the coefficients of the second fundamental form h.

For any real function f on M, the Laplacian operator of M with respect to induced metric is given by

$$\Delta f = -\varepsilon_i \sum_i \left(\tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} f - \tilde{\nabla}_{\nabla_{e_i} e_i} f \right). \tag{2.1}$$

The mean curvature vector H and the Gaussian curvature $K$of M in $\mathbb{E}_s^m$ are defined by

$$H = \frac{1}{n} \sum_{s=n+1}^{m} \sum_{i=1}^{n} \varepsilon_i \varepsilon_s h_{ii}^s e_s \tag{2.2}$$

and

$$K = \sum_{s=n+1}^{m} \varepsilon_s \left(h_{11}^s h_{22}^s - h_{12}^s h_{21}^s \right), \tag{2.3}$$

respectively. We recall that a surface M is called minimal if its mean curvature vector vanishes identically, i.e. $H = 0$. If the mean curvature vector satisfies $DH = 0$, then the surface M is said to have parallel mean curvature vector. Also if Gaussian curvature of M vanishes identically, i.e. $K = 0$, the surface M is called flat.

3 Rotational Surfaces with Pointwise 1-Type Gauss Map in $\mathbb{E}^4_2$

In this section, we consider rotational surfaces of elliptic, hyperbolic and parabolic type in four dimensional pseudo-Euclidean space $\mathbb{E}^4_2$ which are defined by Ganchev and Milousheva in [10] and investigate these rotational surfaces with pointwise 1-type Gauss map.

We denote the standart orthonormal basis of $\mathbb{E}^4_2$ by $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$ where $\epsilon_1 = (1,0,0,0)$, $\epsilon_2 = (0,1,0,0)$, $\epsilon_3 = (0,0,1,0)$ and $\epsilon_4 = (0,0,0,1)$, and $\langle \epsilon_1, \epsilon_1 \rangle = \langle \epsilon_2, \epsilon_2 \rangle = 1$, $\langle \epsilon_3, \epsilon_3 \rangle = \langle \epsilon_4, \epsilon_4 \rangle = -1$.

3.1 Rotational surfaces of elliptic type with pointwise 1-type Gauss map in $\mathbb{E}^4_2$

In this subsection, first we consider the rotational surfaces of elliptic type with harmonic Gauss map.Then, we give a characterization of the flat rotational surfaces of elliptic type with pointwise 1-type Gauss map and obtain a relationship for non-minimal these surfaces with parallel mean curvature vector and pointwise 1-type Gauss map of the first kind.

Rotational surface of elliptic type M_1 is defined by

$$\varphi(t,s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1(s) \\ x_2(s) \\ x_3(s) \\ 0 \end{pmatrix}$$

$$M_1 : \ \varphi(t,s) = (x_1(s), x_2(s), x_3(s)\cos t, x_3(s)\sin t), \tag{3.1}$$

where the surface M_1 is obtained by the rotation of the curve

$$x(s) = (x_1(s), x_2(s), x_3(s), 0)$$

about the two dimensional Euclidean plane span$\{\epsilon_1, \epsilon_2\}$. Let the profile curve of M_1 be unit speed spacelike curve. In that case, $(x_1{}'(s))^2 + (x_2{}'(s))^2 - (x_3{}'(s))^2 = 1$. We suppose that $x_3(s) > 0$. The moving frame field $\{e_1, e_2, e_3, e_4\}$ on M_1 is determined as follows:

$$\begin{aligned} e_1 &= (x_1'(s), x_2'(s), x_3'(s)\cos t, x_3'(s)\sin t), \\ e_2 &= (0,0,-\sin t, \cos t), \\ e_3 &= \frac{1}{\sqrt{1+x_3'(s)^2}}(-x_2'(s), x_1'(s), 0, 0), \\ e_4 &= \frac{1}{\sqrt{1+x_3'(s)^2}}\Big(x_3'(s)x_1'(s), x_3'(s)x_2'(s), (1+x_3'(s)^2)\cos t, \\ &\quad (1+x_3'(s)^2)\sin t\Big), \end{aligned}$$

where e_1, e_2 and e_3, e_4 are tangent vector fields and normal vector fields to M_1, respectively.Then it is easily seen that

$$\langle e_1, e_1\rangle = \langle e_3, e_3\rangle = 1, \ \ \langle e_2, e_2\rangle = \langle e_4, e_4\rangle = -1.$$

We have the dual 1-forms as:

$$\omega_1 = ds \qquad \text{and} \qquad \omega_2 = -x_3(s)dt.$$

After some computations, the components of the second fundamental form and the connection forms are given as follows:

$$\begin{aligned} h_{11}^3 =& - d(s),\ h_{12}^3 = 0,\ h_{22}^3 = 0, \\ h_{11}^4 =& - c(s), \ \ h_{12}^4 = 0, \ \ h_{22}^4 = b(s) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \omega_{12} =& a(s)b(s)\omega_2, \quad \omega_{13} = -d(s)\omega_1, \quad \omega_{14} = -c(s)\omega_1, \\ \omega_{23} =& 0, \quad \omega_{24} = -b(s)\omega_2, \quad \omega_{34} = a(s)d(s)\omega_1. \end{aligned}$$

By taking the covariant derivative with respect to e_1 and e_2 we have

$$\begin{aligned} \tilde{\nabla}_{e_1}e_1 &= -d(s)e_3 + c(s)e_4, \\ \tilde{\nabla}_{e_2}e_1 &= a(s)b(s)e_2, \\ \tilde{\nabla}_{e_1}e_2 &= 0, \\ \tilde{\nabla}_{e_2}e_2 &= a(s)b(s)e_1 - b(s)e_4, \\ \tilde{\nabla}_{e_1}e_3 &= d(s)e_1 - a(s)d(s)e_4, \\ \tilde{\nabla}_{e_2}e_3 &= 0, \\ \tilde{\nabla}_{e_1}e_4 &= c(s)e_1 - a(s)d(s)e_3, \\ \tilde{\nabla}_{e_2}e_4 &= b(s)e_2, \end{aligned} \tag{3.3}$$

where

$$a(s) = \frac{x_3'(s)}{\sqrt{1+(x_3{}')^2}}, \tag{3.4}$$

$$b(s) = \frac{\sqrt{1+(x_3{}')^2}}{x_3(s)}, \tag{3.5}$$

$$c(s) = \frac{x_3''(s)}{\sqrt{1+(x_3{}')^2}}, \tag{3.6}$$

$$d(s) = \frac{x_1''(s)x_2'(s) - x_2''(s)x_1'(s)}{\sqrt{1+(x_3{}')^2}}. \tag{3.7}$$

By using (2.2), (2.3) and (3.2), the mean curvature vector and Gaussian curvature of the surface M_1 are obtained as:

$$H = \frac{1}{2}\left(-d(s)e_3 + \left(c(s) + b\left(s\right)\right)e_4\right) \tag{3.8}$$

and

$$K = c(s)b(s), \tag{3.9}$$

respectively.

By using (2.1) and (3.3), we find the Laplacian of the Gauss map of M_1 as :

$$\Delta G = L(s)(e_1 \wedge e_2) + M(s)(e_2 \wedge e_3) + N(s)(e_2 \wedge e_4), \tag{3.10}$$

where

$$L(s) = d^2(s) - b^2(s) - c^2(s), \tag{3.11}$$

$$M(s) = d'(s) + a(s)d(s)(b(s) + c(s)), \tag{3.12}$$

$$N(s) = b'(s) + c'(s) + a(s)d^2(s). \tag{3.13}$$

Theorem 3.1. *Let M_1 be rotation surface of elliptic type given by the parametrization (3.1). If M_1 has harmonic Gauss map then it has constant Gaussian curvature.*

Proof. Let the Gauss map of M_1 be harmonic, i.e., $\Delta G = 0$. So, from (3.10), (3.11), (3.12) and (3.13) we have

$$\begin{aligned} d^2(s) - b^2(s) - c^2(s) &= 0, \\ d'(s) + a(s)d(s)(b(s) + c(s)) &= 0, \\ b'(s) + c'(s) + a(s)d^2(s) &= 0. \end{aligned} \tag{3.14}$$

By multiplying both sides of the second equation of (3.14) with $d(s)$ and using the third equation of (3.14) we have

$$d(s)d'(s) - b(s)b'(s) - c(s)c'(s) = (b(s)c(s))'. \tag{3.15}$$

By differeniating the first equation of (3.14) with respect to s and using (3.15), we have that $b(s)c(s)$ =constant.Hence, from (3.9) we get $K = K_0$ =constant. ☐

Theorem 3.2. *Let M_1 be the flat rotational surface of elliptic type given by the parametrization (3.1). Then M_1 has a pointwise 1-type Gauss map if and only if the profile curve of M_1 is characterized by one of the following way:*

i)

$$\begin{aligned} x_1(s) &= -\frac{1}{\delta_1}\sin(-\delta_1 s + \delta_2) + \delta_4, \\ x_2(s) &= \frac{1}{\delta_1}\cos(-\delta_1 s + \delta_2) + \delta_4, \\ x_3(s) &= \delta_3, \end{aligned} \tag{3.16}$$

where δ_1, δ_2, δ_3 and δ_4 are real constants and the Gauss map of M_1 satisfies (1.1) for $f = \delta_1^2 - \frac{1}{\delta_3^2}$ and $C = 0$. If $\delta_1\delta_3 = \pm 1$ then the function f becomes zero and it implies that the Gauss map is harmonic.

ii)

$$\begin{aligned} x_1(s) =& \int \left(1+\lambda_1^2\right)^{\frac{1}{2}} \cos\left(-\frac{\lambda_3}{\lambda_1\left(1+\lambda_1^2\right)^{\frac{1}{2}}} \ln(\lambda_1 s+\lambda_2)+\lambda_4\right) ds, \\ x_2(s) =& \int \left(1+\lambda_1^2\right)^{\frac{1}{2}} \sin\left(-\frac{\lambda_3}{\lambda_1\left(1+\lambda_1^2\right)^{\frac{1}{2}}} \ln(\lambda_1 s+\lambda_2)+\lambda_4\right) ds, \\ x_3(s) =& \lambda_1 s+\lambda_2, \end{aligned} \tag{3.17}$$

where λ_1, λ_2, λ_3 and λ_4 are real constants and the Gauss map of M_1 satisfies (1.1) for $f(s)=\frac{1}{(\lambda_1 s+\lambda_2)^2}\left(\frac{\lambda_3^2}{1+\lambda_1^2}-1\right)$ and $C=\lambda_1^2 e_1\wedge e_2+\lambda_1\left(1+\lambda_1^2\right)^{\frac{1}{2}} e_2\wedge e_4$.

Proof. We suppose that M_1 has pointwise 1-type Gauss map. By using (1.1) and (3.10), we get

$$\begin{aligned} -f+f\left\langle C,e_1\wedge e_2\right\rangle &= -L(s), \\ f\left\langle C,e_2\wedge e_3\right\rangle &= -M(s), \\ f\left\langle C,e_2\wedge e_4\right\rangle &= N(s) \end{aligned} \tag{3.18}$$

and

$$\left\langle C,e_1\wedge e_3\right\rangle=\left\langle C,e_1\wedge e_4\right\rangle=\left\langle C,e_3\wedge e_4\right\rangle=0. \tag{3.19}$$

By taking the derivatives of all equations in (3.19) with respect to e_2 and using (3.18) we obtain

$$\begin{aligned} a(s)N(s)-L(s)+f &= 0, \\ a(s)M(s) &= 0, \\ M(s) &= 0, \end{aligned} \tag{3.20}$$

respectively. From above equations, we have two cases. One of them is $a(s)=0$, $M(s)=0$ and the other is $a(s)\neq 0$, $M(s)=0$. Firstly, we suppose that $a(s)=0$ and $M(s)=0$. By using (3.4), we have that $x_3(s)=\delta_3$=constant. It implies that $c(s)=0$, $b\left(s\right)=\frac{1}{\delta_3}$ and M_1 is flat. Since the profile curve x is spacelike curve which is parameterized by arc-length, we can put

$$\begin{aligned} x_1'(s) &= \cos\delta(s) \text{ (or resp. } \sin\delta(s)), \\ x_2'(s) &= \sin\delta(s) \text{ (or resp. } \cos\delta(s)), \end{aligned} \tag{3.21}$$

where δ is smooth angle function. Without loss of generality we assume that

$$x_1'(s)=\cos\delta\left(s\right) \text{ and } x_2'(s)=\sin\delta\left(s\right)$$

We can do similar computations for the another case, too. By using third equation of (3.20) and (3.12) we obtain that

$$d\left(s\right)=\delta_1,\ \delta_1 \text{ is non zero constant.} \tag{3.22}$$

On the other hand by using (3.7), (3.21) and (3.22) we get

$$\delta\left(s\right)=-\delta_1 s+\delta_2, \tag{3.23}$$

where δ_1, δ_2 are real constants. Then by substituting (3.23) into (3.21) and taking the integral we have the equation (3.16). Also the Laplacian of the Gauss map of M_1 with the equations $a(s)=0$, $b\left(s\right)=\frac{1}{\delta_3}$, $c(s)=0$ and $d\left(s\right)=\delta_1$ is found as $\Delta G=\left(\delta_1^2-\frac{1}{\delta_3^2}\right)G$

Now we suppose that $a(s)\neq 0$ and $M(s)=0$. Since the surface M_1 is flat, i.e., $K=0$. By using (3.9) we have that $c(s)=0$. From (3.6) we get

$$x_3(s)=\lambda_1 s+\lambda_2 \tag{3.24}$$

for some constants $\lambda_1\neq 0$ and λ_2. In that case by using (3.4), (3.5) and (3.24) we have

$$a(s)=\frac{\lambda_1}{(1+\lambda_1^2)^{\frac{1}{2}}} \tag{3.25}$$

and

$$b(s)=\frac{\left(1+\lambda_1^2\right)^{\frac{1}{2}}}{\lambda_1 s+\lambda_2}. \tag{3.26}$$

Let consider that $M(s)=0$ with $c(s)=0$. In that case from (3.12), we obtain that

$$d'\left(s\right)+a(s)b(s)d(s)=0 \tag{3.27}$$

By using (3.25), (3.26) and (3.27) we have

$$d(s)=\frac{\lambda_3}{\lambda_1 s+\lambda_2}, \tag{3.28}$$

where λ_3 is constant of integration. On the other hand, Since the profile curve x is spacelike curve which is parameterized by arc-length, we can put

$$\begin{aligned} x_1'(s) &= \left(1+\lambda_1^2\right)^{\frac{1}{2}}\cos\lambda\left(s\right), \\ x_2'(s) &= \left(1+\lambda_1^2\right)^{\frac{1}{2}}\sin\lambda\left(s\right), \end{aligned} \tag{3.29}$$

where λ is smooth angle function. By differentiating (3.29). we obtain

$$\begin{aligned} x_1''(s) &= -\left(1+\lambda_1^2\right)^{\frac{1}{2}}\sin\lambda\left(s\right)\lambda'\left(s\right), \\ x_2''(s) &= \left(1+\lambda_1^2\right)^{\frac{1}{2}}\cos\lambda\left(s\right)\lambda'\left(s\right). \end{aligned} \tag{3.30}$$

By using (3.7), (3.24), (3.29) and (3.30), we get

$$d(s)=-\left(1+\lambda_1^2\right)^{\frac{1}{2}}\lambda'\left(s\right). \tag{3.31}$$

By combining (3.28) and (3.31) we obtain

$$\lambda\left(s\right)=-\frac{\lambda_3}{\lambda_1\left(1+\lambda_1^2\right)^{\frac{1}{2}}}\ln(\lambda_1 s+\lambda_2)+\lambda_4. \tag{3.32}$$

So by substituting (3.32) into (3.29), we get

$$
\begin{aligned}
x_1(s) &= \int \left(1+\lambda_1^2\right)^{\frac{1}{2}} \cos\left(-\frac{\lambda_3}{\lambda_1\left(1+\lambda_1^2\right)^{\frac{1}{2}}}\ln(\lambda_1 s+\lambda_2)+\lambda_4\right) ds,\\
x_2(s) &= \int \left(1+\lambda_1^2\right)^{\frac{1}{2}} \sin\left(-\frac{\lambda_3}{\lambda_1\left(1+\lambda_1^2\right)^{\frac{1}{2}}}\ln(\lambda_1 s+\lambda_2)+\lambda_4\right) ds,
\end{aligned}
$$

Conversely, the surface M_1 whose the profil curve given by (3.17) is pointwise 1-type Gauss map for

$$f(s)=\frac{1}{\left(\lambda_1 s+\lambda_2\right)^2}\left(\frac{\lambda_3^2}{1+\lambda_1^2}-1\right)$$

and

$$C=\lambda_1^2 e_1\wedge e_2+\lambda_1\left(1+\lambda_1^2\right)^{\frac{1}{2}} e_2\wedge e_4.$$

□

Theorem 3.3. *A non- minimal rotational surfaces of elliptic type M_1 defined by (3.1) has pointwise 1-type Gauss map of the first kind if and only if the mean curvature vector of M_1 is parallel .*

Proof. From (3.8) we have that $H=\frac{1}{2}\left(-d(s)e_3+\left(c(s)+b\left(s\right)\right)e_4\right)$. Let the mean curvature vector of M_1 be parallel, i.e., $DH=0$. Then we get

$$D_{e_1}H=\frac{1}{2}\left(-M(s)e_3+N(s)e_4\right)=0.$$

In this case we obtain that $M(s)=N(s)=0$. From (3.10), we have that $\Delta G=L(s)e_1\wedge e_2$.

Conversely, if M_1 has pointwise 1-type Gauss map of the first kind then from (3.10) we get $M(s)=N(s)=0$ and it implies that M_1 has parallel mean curvature vector. □

Corollary 3.4. *If rotational surfaces of elliptic type M_1 given by (3.1) is minimal then it has pointwise 1-type Gauss map of the first kind.*

3.2 Rotational surfaces of hyperbolic type with pointwise 1-type Gauss map in $\mathbb{E}_2^4$

In this subsection, first we consider rotational surfaces of hyperbolic type with harmonic Gauss map. Moreover, we obtain a characterization of flat rotational surfaces of hyperbolic type with pointwise 1-type Gauss map and give a relationship for non-minimal these surfaces with parallel mean curvature vector and pointwise 1-type Gauss map of the first kind. The proofs of theorems in this subsection are similar the proofs of theorems in previous section so we give the theorems as without proof.

Rotational surface of hyperbolic type M_2 is defined by

$$\varphi(t,s) = \begin{pmatrix} \cosh t & 0 & \sinh t & 0 \\ 0 & 1 & 0 & 0 \\ \sinh t & 0 & \cosh t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(s) \\ x_2(s) \\ 0 \\ x_4(s) \end{pmatrix}$$

$$M_2: \ \varphi(t,s) = (x_1(s)\cosh t, x_2(s), x_1(s)\sinh t, x_4(s)), \tag{3.33}$$

where the surface M_2 is obtained by the rotation of the curve

$$x(s) = (x_1(s), x_2(s), 0, x_4(s))$$

about the two dimensional Euclidean plane spanned by ϵ_2 and ϵ_4. Let the profile curve of M_2 be unit speed spacelike curve. In that case $(x_1{}'(s))^2 + (x_2{}'(s))^2 - (x_4{}'(s))^2 = 1$. We assume that $x_1(s) > 0$. The moving frame field $\{e_1, e_2, e_3, e_4\}$ on M_2 is choosen as follows:

$$\begin{aligned} e_1 &= (x_1'(s)\cosh t, x_2'(s), x_1'(s)\sinh t, x_4'(s)), \\ e_2 &= (\sinh t, 0, \cosh t, 0), \\ e_3 &= \frac{1}{\sqrt{\varepsilon\,(x_1'(s)^2 - 1)}}(0, x_4'(s), 0, x_2'(s)), \\ e_4 &= \frac{1}{\sqrt{\varepsilon(x_1'(s)^2 - 1)}}\Big(\left(x_1'(s)^2 - 1\right)\cosh t, -x_1'(s)x_2'(s), \left(x_1'(s)^2 - 1\right)\sinh t, \\ &\quad -x_1'(s)x_4'(s)\Big), \end{aligned}$$

where e_1, e_2 and e_3, e_4 are tangent vector fields and normal vector fields to M_2, respectively and ε is signature of $(x_1{}')^2 - 1$. If $(x_1{}')^2 - 1$ is positive (resp. negative) then $\varepsilon = 1$ (resp. $\varepsilon = -1$). It is easily seen that

$$\langle e_1, e_1 \rangle = -\langle e_2, e_2 \rangle = 1, \ \ \langle e_3, e_3 \rangle = -\langle e_4, e_4 \rangle = \varepsilon.$$

we have the dual 1-forms as:

$$\omega_1 = ds \qquad \text{and} \qquad \omega_2 = -x_1(s)dt.$$

After some computations, components of the second fundamental form and the connection forms are obtained by:

$$\begin{aligned} h_{11}^3 &= d(s), \ h_{12}^3 = 0, \ h_{22}^3 = 0, \\ h_{11}^4 &= c(s), \ \ h_{12}^4 = 0, \ \ h_{22}^4 = -\varepsilon b(s) \end{aligned} \tag{3.34}$$

and

$$\begin{aligned} \omega_{12} &= a(s)b(s)\omega_2, \quad \omega_{13} = d(s)\omega_1, \quad \omega_{14} = c(s)\omega_1, \\ \omega_{23} &= 0, \quad \omega_{24} = \varepsilon b(s)\omega_2, \quad \omega_{34} = a(s)d(s)\omega_1. \end{aligned}$$

Differentiating covariantly with respect to e_1 and e_2 we get

$$\begin{aligned}
\tilde{\nabla}_{e_1} e_1 &= \varepsilon d(s) e_3 - \varepsilon c(s) e_4 \\
\tilde{\nabla}_{e_2} e_1 &= a(s) b(s) e_2 \\
\tilde{\nabla}_{e_1} e_2 &= 0 \\
\tilde{\nabla}_{e_2} e_2 &= a(s) b(s) e_1 + b(s) e_4 \\
\tilde{\nabla}_{e_1} e_3 &= -d(s) e_1 - \varepsilon a(s) d(s) e_4 \\
\tilde{\nabla}_{e_2} e_3 &= 0 \\
\tilde{\nabla}_{e_1} e_4 &= -c(s) e_1 - \varepsilon a(s) d(s) e_3 \\
\tilde{\nabla}_{e_2} e_4 &= -\varepsilon b(s) e_2
\end{aligned} \tag{3.35}$$

where

$$a(s) = \frac{x_1'(s)}{\sqrt{\varepsilon \left((x_1{'})^2 - 1 \right)}},$$

$$b(s) = \frac{\sqrt{\varepsilon \left((x_1{'})^2 - 1 \right)}}{x_1(s)},$$

$$c(s) = \frac{x_1''(s)}{\sqrt{\varepsilon \left((x_1{'})^2 - 1 \right)}},$$

$$d(s) = \frac{x_2''(s) x_4'(s) - x_4''(s) x_2'(s)}{\sqrt{\varepsilon \left((x_1{'})^2 - 1 \right)}}.$$

By using (2.2), (2.3) and (3.34), the mean curvature vector and Gaussian curvature of the surface M_2 are obtained as follows:

$$H = \frac{1}{2} \left(\varepsilon d(s) e_3 - \varepsilon \left(c(s) + \varepsilon b\left(s\right) \right) e_4 \right)$$

and

$$K = c(s) b\left(s\right),$$

respectively.

By using (2.1) and (3.35) , we find the Laplacian of the Gauss map of M_2 as:

$$\Delta G = L(s) \left(e_1 \wedge e_2 \right) + M(s) \left(e_2 \wedge e_3 \right) + N(s) \left(e_2 \wedge e_4 \right),$$

where

$$L(s) = \varepsilon \left(d^2(s) - c^2\left(s\right) - b^2\left(s\right) \right),$$

$$M(s) = \varepsilon \left(d'\left(s\right) + \varepsilon a(s) d(s) \left(c(s) + \varepsilon b\left(s\right) \right) \right),$$

$$N(s) = -\varepsilon \left(c'(s) + \varepsilon b'(s) + \varepsilon a(s) d^2(s) \right).$$

Theorem 3.5. *Let M_2 be rotation surface of hyperbolic type given by the parameterization (3.33). If M_2 has Gauss map harmonic then it has constant Gaussian curvatrure.*

Theorem 3.6. *Let M_2 be flat rotation surface of hyperbolic type given by the parameterization (3.33). Then M_2 has pointwise 1-type Gauss map if and only if the profile curve of M_2 is characterized in one of the following way:*

i)

$$\begin{aligned} x_1(s) &= \delta_1, \\ x_2(s) &= -\frac{1}{\delta_2} \sinh\left(-\delta_2 s + \delta_3\right) + \delta_4, \\ x_4(s) &= -\frac{1}{\delta_2} \cosh\left(-\delta_2 s + \delta_3\right) + \delta_4, \end{aligned}$$

where δ_1, δ_2, δ_3 and δ_4 are real constants and the Gauss map G satisfies (1.1) for $f = \frac{1}{\delta_1^2} - \delta_2^2$ and $C = 0$. If $\delta_1\delta_2 = \pm 1$ then the function f becomes zero and it implies that the Gauss map is harmonic.

ii)

$$\begin{aligned} x_1(s) &= \lambda_1 s + \lambda_2, \\ x_2(s) &= \int \left(\lambda_1^2 - 1\right)^{\frac{1}{2}} \sinh\left(\frac{\lambda_3}{\lambda_1 \left(\lambda_1^2 - 1\right)^{\frac{1}{2}}} \ln(\lambda_1 s + \lambda_2) + \lambda_4\right) ds, \\ x_4(s) &= \int \left(\lambda_1^2 - 1\right)^{\frac{1}{2}} \cosh\left(\frac{\lambda_3}{\lambda_1 \left(\lambda_1^2 - 1\right)^{\frac{1}{2}}} \ln(\lambda_1 s + \lambda_2) + \lambda_4\right) ds, \end{aligned}$$

where λ_1, λ_2, λ_3 and λ_4 are real constants and without loss of generality we suppose that $\lambda_1^2 - 1 > 0$. Morever the Gauss map G satisfies (1.1) for the function $f(s) = \frac{1}{(\lambda_1 s + \lambda_2)^2}\left(1 - \frac{\lambda_3^2}{\lambda_1^2 - 1}\right)$ and $C = -\lambda_1^2 e_1 \wedge e_2 + \lambda_1 \left(\lambda_1^2 - 1\right)^{\frac{1}{2}} e_2 \wedge e_4$.

Theorem 3.7. *A non- minimal rotational surfaces of hyperbolic type M_2 defined by (3.33) has pointwise 1-type Gauss map of the first kind if and only if M_2 has parallel mean curvature vector*

Corollary 3.8. *If rotational surfaces of hyperbolic type M_2 given by (3.33) is minimal then it has pointwise 1-type Gauss map of the first kind.*

3.3 Rotational surfaces of parabolic type with pointwise 1-type Gauss map in $\mathbb{E}^4_2$

In this subsection, we study rotational surfaces of parabolic type with pointwise 1-type Gauss map. We show that flat rotational surface of parabolic type has pointwise 1-type Gauss map if and only if its Gauss map is harmonic. Also we conclude that flat rotational surface of parabolic type has harmonic Gauss map if and only if it has parallel mean curvature vector.

We consider the pseudo-orthonormal base $\{\epsilon_1, \xi_2, \xi_3, \epsilon_4\}$ of $\mathbb{E}^4_2$ such that $\xi_2 = \frac{\epsilon_2+\epsilon_3}{\sqrt{2}}$, $\xi_3 = \frac{-\epsilon_2+\epsilon_3}{\sqrt{2}}$ $\langle \xi_2, \xi_2\rangle = \langle \xi_3, \xi_3\rangle = 0$ and $\langle \xi_2, \xi_3\rangle = -1$. Let consider α spacelike curve is given by

$$x\,(s) = x_1(s)\epsilon_1 + x_2(s)\epsilon_2 + x_3(s)\epsilon_3$$

or we can express x according to pseudo-orthonormal base $\{\epsilon_1, \xi_2, \xi_3, \epsilon_4\}$ as follows:

$$x\,(s) = x_1(s)\epsilon_1 + p(s)\xi_2 + q(s)\xi_3,$$

where $p(s) = \frac{x_2(s)+x_3(s)}{\sqrt{2}}$ and $q(s) = \frac{-x_2(s)+x_3(s)}{\sqrt{2}}$. The rotational surface of parabolic type M_3 is defined by

$$M_3 : \varphi\,(t, s) = x_1(s)\epsilon_1 + p(s)\xi_2 + (-t^2 p(s) + q(s))\xi_3 + \sqrt{2}tp(s)\epsilon_4, \qquad (3.36)$$

We suppose that x is parameterized by arc-length, that is, $({x_1}'(s))^2 - 2p'(s)q'(s) = 1$. Now we can give a moving orthonormal frame $\{e_1, e_2, e_3, e_4\}$ for M_3 as follows:

$$\begin{aligned}
e_1 &= {x_1}'(s)\epsilon_1 + p'(s)\xi_2 + (-t^2 p'(s) + q'(s))\xi_3 + \sqrt{2}tp'(s)\epsilon_4,\\
e_2 &= -\sqrt{2}t\xi_3 + \epsilon_4,\\
e_3 &= \epsilon_1 + \frac{{x_1}'(s)}{p'(s)}\xi_3,\\
e_4 &= {x_1}'(s)\epsilon_1 + p'(s)\xi_2 + (\frac{1}{p'(s)} + q'(s) - t^2 p'(s))\xi_3 + \sqrt{2}tp'(s)\epsilon_4,
\end{aligned}$$

where $p'(s)$ is non zero. Then it is easily seen that

$$\langle e_1, e_1\rangle = \langle e_3, e_3\rangle = 1, \;\; \langle e_2, e_2\rangle = \langle e_4, e_4\rangle = -1.$$

We have the dual 1-forms as:

$$\omega_1 = ds \qquad \text{and} \qquad \omega_2 = -\sqrt{2}p\,(s)\,dt.$$

Also we obtain components of the second fundamental form and the connection forms as:

$$\begin{aligned}
h^3_{11} &= c(s),\;\; h^3_{12} = 0,\;\; h^3_{22} = 0,\\
h^4_{11} &= -\,b(s),\quad h^4_{12} = 0,\quad h^4_{22} = a(s)
\end{aligned} \qquad (3.37)$$

and

$$\begin{aligned}
\omega_{12} &= a(s)\omega_2, \quad \omega_{13} = c(s)\omega_1, \quad \omega_{14} = -b(s)\omega_1,\\
\omega_{23} &= 0, \quad \omega_{24} = -a(s)\omega_2, \quad \omega_{34} = -c(s)\omega_1.
\end{aligned}$$

Then, by taking the covariant derivatives with respect to e_1 and $e_2, weget$ as follows:

$$\begin{aligned}
\tilde{\nabla}_{e_1} e_1 &= c(s)e_3 + b(s)e_4, \\
\tilde{\nabla}_{e_2} e_1 &= a(s)e_2, \\
\tilde{\nabla}_{e_1} e_2 &= 0, \\
\tilde{\nabla}_{e_2} e_2 &= a(s)e_1 - a(s)e_4, \\
\tilde{\nabla}_{e_1} e_3 &= -c(s)e_1 + c(s)e_4, \\
\tilde{\nabla}_{e_2} e_3 &= 0, \\
\tilde{\nabla}_{e_1} e_4 &= b(s)e_1 + c(s)e_3, \\
\tilde{\nabla}_{e_2} e_4 &= a(s)e_2,
\end{aligned} \tag{3.38}$$

where

$$a(s) = \frac{p'(s)}{p(s)}, \tag{3.39}$$

$$b(s) = \frac{p''(s)}{p'(s)}, \tag{3.40}$$

$$c(s) = \frac{x_1''(s)p'(s) - p''(s)x_1'(s)}{p'(s)}. \tag{3.41}$$

By using (2.2), (2.3) and (3.37), the mean curvature vector and Gaussian curvature of the surface M_3 are obtained as follows:

$$H = \frac{1}{2}\left(c(s)e_3 + \left(a(s) + b\left(s\right)\right)e_4\right) \tag{3.42}$$

and

$$K = a(s)b\left(s\right), \tag{3.43}$$

respectively.

By using (2.1) and (3.38), we find the Laplacian of the Gauss map of M_3 by

$$\Delta G = L(s)\left(e_1 \wedge e_2\right) + M(s)\left(e_2 \wedge e_3\right) + N(s)\left(e_2 \wedge e_4\right), \tag{3.44}$$

where

$$L(s) = c^2(s) - a^2\left(s\right) - b^2\left(s\right), \tag{3.45}$$

$$M(s) = c'\left(s\right) + c(s)(a(s) + b(s)), \tag{3.46}$$

$$N(s) = c^2(s) + a'\left(s\right) + b'\left(s\right). \tag{3.47}$$

Theorem 3.9. *Let M_3 be flat rotation surface of parabolic type given by the parameterization (3.36). Then M_3 has pointwise 1-type Gauss map if and only if the profile curve of M_3 is given by*

$$\begin{aligned}
x_1(s) &= \frac{\varepsilon}{\mu_1}\left(\ln(\mu_1 s + \mu_2)(\mu_1 s + \mu_2)\right) + \left(\mu_4 - \varepsilon\right)s + \mu_5, \\
p(s) &= \mu_1 s + \mu_2, \\
q(s) &= \frac{1}{2\mu_1}\int \left(\left(\varepsilon \ln\left(\mu_1 s + \mu_2\right) + \mu_4\right)^2 - 1\right) ds,
\end{aligned}$$

where μ_1, μ_2, μ_4, μ_5 real constants. Morever the surface M_3 has harmonic Gauss map for $f = 0$.

Proof. We suppose that M_3 has pointwise 1-type Gauss map. In that case the Gauss map of M_3 satisfies (1.1). By using (1.1) and (3.44), we get

$$\begin{aligned} -f + f\langle C, e_1 \wedge e_2\rangle &= -L(s), \\ f\langle C, e_2 \wedge e_3\rangle &= -M(s), \\ f\langle C, e_2 \wedge e_4\rangle &= N(s) \end{aligned} \tag{3.48}$$

and

$$\langle C, e_1 \wedge e_3\rangle = \langle C, e_1 \wedge e_4\rangle = \langle C, e_3 \wedge e_4\rangle = 0. \tag{3.49}$$

By taking the derivatives of all equations in (3.49) with respect to e_2 and using (3.48) we obtain

$$\begin{aligned} L(s) - N(s) &= f, \\ M(s) &= 0, \end{aligned} \tag{3.50}$$

respectively. Since the surface M_3 is flat, i.e., $K = 0$ from (3.43) we have that $b(s) = 0$. From (3.40) we obtain that

$$p(s) = \mu_1 s + \mu_2 \tag{3.51}$$

for some constants $\mu_1 \neq 0$ and μ_2. By using (3.39) and (3.51) we have that

$$a(s) = \frac{\mu_1}{\mu_1 s + \mu_2}. \tag{3.52}$$

If we consider $M(s) = 0$ with the equations $b(s) = 0$ and $a(s) = \frac{\mu_1}{\mu_1 s + \mu_2}$, from (3.46) we get

$$c(s) = \frac{\mu_3}{\mu_1 s + \mu_2}. \tag{3.53}$$

On the other hand, by using the first equation of (3.50), (3.45), (3.47), (3.52) and (3.53) we obtain that $f = 0$. It means that $L(s) = N(s) = 0$ and we have

$$\mu_3 = \varepsilon\mu_1, \ \varepsilon = \pm 1.$$

If we consider (3.41), (3.51) and (3.53) we get

$$x_1(s) = \frac{\varepsilon}{\mu_1}\left(\ln(\mu_1 s + \mu_2)(\mu_1 s + \mu_2)\right) + (\mu_4 - \varepsilon)\, s + \mu_5, \tag{3.54}$$

where μ_4, μ_5 are constants of integration. Since x is unit speed spacelike curve we get

$$q'(s) = \frac{(x_1{}'(s))^2 - 1}{2p'(s)}. \tag{3.55}$$

By substituting (3.51) and (3.54) into (3.55) we obtain

$$q(s) = \frac{1}{2\mu_1}\int \left((\varepsilon \ln (\mu_1 s + \mu_2) + \mu_4)^2 - 1\right) ds.$$

This completes the proof. □

Theorem 3.10. *Let M_3 be flat rotational surfaces of parabolic type given by (3.36). M_3 has harmonic Gauss map if and only if its mean curvature vector is parallel.*

Proof. We suppose that M_3 has parallel mean curvature vector, i.e., $DH = 0$. From (3.42) we have that

$$D_{e_1}H = \frac{1}{2}\left(M(s)e_3 + N(s)e_4\right) = 0.$$

In this case we obtain that $M(s) = N(s) = 0$. Since M_3 is a flat surface, from the previous theorem we have

$$b(s) = 0 \text{ and } a(s) = \frac{\mu_1}{\mu_1 s + \mu_2}.$$

By considering the equation $M(s) = 0$ with above equations and using (3.46) we get

$$c(s) = \frac{\mu_3}{\mu_1 s + \mu_2},$$

where μ_3 is the constant of integration. It implies that $L(s) = 0$. Hence we obtain that Gauss map of M_3 is harmonic .

Conversely, if M_3 is harmonic then it is easily seen that $DH = 0$. □

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