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# On Slant Curves with Pseudo-Hermitian $C$-parallel Mean Curvature Vector Fields

**Cihan Özgür**

Cihan Özgür: Balıkesir University, Department of Mathematics, 10145, Çağış, Balıkesir, Turkey, e-mail: cozgur@balikesir.edu.tr

**Abstract.** We study pseudo-Hermitian $C$-parallel and $C$-proper slant curves in contact metric 3-manifolds. As an application, we give two examples of pseudo-Hermitian Legendre circle and pseudo-Hermitian slant helix in Sasakian Heisenberg group.
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## 1 Introduction

In [7], Chen defined biharmonic submanifold as a Riemannian submanifold with vanishing Laplacian of mean curvature vector field $\Delta H$. Curves in a Euclidean space satisfying the condition $\Delta^{\perp} H = \lambda H$ were classified in [2], by Barros and Garay, where $\Delta^{\perp}$ denotes the Laplacian of the curve in the normal bundle and $\lambda$ is a real valued function. In the real space form, the classification of curves satisfying $\Delta H = \lambda H$ and $\Delta^{\perp} H = \lambda H$ were given in [1], by Arroyo, Barros and Garay.

A curve in a contact metric manifold is said to be *slant* [9], if its tangent vector field has a constant angle with the Reeb vector field. In particular, if the contact angle is equal to $\frac{\pi}{2}$, then the curve is called a *Legendre curve*. In [8], Cho and Lee studied slant curves in pseudo-Hermitian contact 3-manifolds. Legendre curves with pseudo-Hermitian parallel mean curvature vector field, pseudo-Hermitian proper mean curvature vector field and pseudo-Hermitian proper mean curvature vector field in the normal bundle in contact pseudo-Hermitian 3-manifolds were studied by Lee in [12]. In [14], the present author and Güvenç studied slant curves with pseudo-Hermitian parallel mean curvature vector field, pseudo-Hermitian proper mean curvature vector field and pseudo-Hermitian proper mean curvature vector field in the normal bundle in contact pseudo-Hermitian 3-manifolds. The notions of $C$-parallel and $C$-proper curves in the tangent and normal bundles were introduced by Lee, Suh and Lee in [13]. A curve in an almost contact metric manifold is defined to be *$C$-parallel* if $\nabla_T H = \lambda\xi$, *$C$-proper* if $\Delta H = \lambda\xi$, *$C$-parallel in the normal bundle* if $\nabla^{\perp}_T H = \lambda\xi$, *$C$-proper in the normal bundle* if $\Delta^{\perp} H = \lambda\xi$, where $T$ is the unit tangent vector field of the curve and $\lambda$ is a differentiable function along the curve. In [13], Lee, Suh and Lee studied $C$-parallel and $C$-proper slant curves in Sasakian 3-manifolds. $C$-parallel and $C$-proper slant curves in trans-Sasakian manifolds were studied in [15], by Güvenç and the present author. On the other hand, slant and Legendre curves in Bianchi-Cartan-Vranceanu geometry were studied by Călin and Crasmareanu in [6]. Slant curves in normal almost contact geometry were studied in [5].

Motivated by the above studies, in the present paper, we study pseudo-Hermitian $C$-parallel and $C$-proper slant curves in contact metric 3-manifolds. We give two examples of pseudo-Hermitian Legendre circle and pseudo-Hermitian slant helix in Sasakian Heisenberg group.

## 2 Preliminaries

Let $M$ be a $(2n+1)$-dimensional manifold. $M$ is called a *contact manifold* [3] if there exists a global 1-form $\eta$ such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on $M$. Given a contact form $\eta$, there exists a unique vector field $\xi$, the *characteristic vector field*, which satisfies $\eta(\xi) = 1$ and $d\eta(X, \xi) = 0$ for any vector field $X$ on $M$. There exists an associated Riemannian metric $g$ and a $(1,1)$-type tensor field $\varphi$ satisfying

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \varphi Y), \tag{2.1}$$

for all $X, Y \in \chi(M)$. From (2.1), it is easy to see that

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.2}$$

A Riemannian manifold equipped with the structure tensors $(\varphi, \xi, \eta, g)$ satisfying (2.1) is called a *contact metric manifold*. It is denoted by $M = \{M, \varphi, \xi, \eta, g\}$. The operator $h$ is defined by $h = \frac{1}{2} L_\xi \varphi$, where $L_\xi$ is the Lie differentiation operator in the characteristic direction $\xi$. From the definition of $h$, it is easy to see that $h$ is symmetric and satisfies the following equations (see [3], page 67):

$$h\xi = 0, \quad h\varphi = -\varphi h, \quad \nabla_X \xi = -\varphi X - \varphi h X, \tag{2.3}$$

where $\nabla$ denotes the Levi-Civita connection.

For a $(2n+1)$-dimensional contact metric manifold $M = \{M, \varphi, \xi, \eta, g\}$, the almost complex structure $J$ on $M \times \mathbb{R}$ is defined by

$$J(X, f\tfrac{d}{dt}) = (\varphi X - f\xi, \eta(X)\tfrac{d}{dt}), \tag{2.4}$$

where $X$ is a vector field tangent to $M$, $t$ is the coordinate function of $\mathbb{R}$ and $f$ is a $C^\infty$ function on $M \times \mathbb{R}$. If $J$ is integrable then the contact metric manifold $M$ is called a *Sasakian manifold* [3].

For a $(2n+1)$-dimensional contact metric manifold $M = \{M, \varphi, \xi, \eta, g\}$ provides a splitting of the tangent bundle

$$TM = Ker(\varphi) \oplus \mathrm{Im}(\varphi)$$

and the restriction $J = \varphi\mid_D$ defines an almost complex structure on $D = \mathrm{Im}(\varphi)$. There is a well-known concept of almost $CR$-structure as follows: Let $M$ be a $(2n+s)$-dimensional smooth manifold. Let $\mathcal{D}$ be a smooth distribution on $M$ of real dimension $2n$ and $J$ a $(1,1)$-tensor field on $M$ such that

$$J^2X = -X, \quad X \in \mathcal{D}.$$

Then $(\mathcal{D}, J)$ is called almost complex distribution (or an almost $CR$-structure). Then $M$ is an *almost $CR$-manifold* (or *a contact strongly pseudo-convex pseudo-Hermitian manifold*) [3].

The Tanaka-Webster connection $\widehat{\nabla}$ (or the pseudo-Hermitian connection) ([16], [18]) on a contact strongly pseudo-convex pseudo-Hermitian manifold $M$ is defined by

$$\widehat{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi$$

for all $X, Y \in \chi(M)$. By the use of (2.3), $\widehat{\nabla}$ can be rewritten as

$$\widehat{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + \eta(Y)(\varphi X + \varphi hX) - g(\varphi X + \varphi hX, Y)\xi. \tag{2.5}$$

From (2.5), the torsion of the Tanaka-Webster connection $\widehat{\nabla}$ is

$$\widehat{T}(X,Y) = 2g(X, \varphi Y)\xi + \eta(Y)\varphi hX - \eta(X)\varphi hY. \tag{2.6}$$

If $M$ is a Sasakian manifold, since $h = 0$, then the equations (2.5) and (2.6) turn into

$$\begin{gathered} \widehat{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + \eta(Y)\varphi X - g(\varphi X, Y)\xi, \\ \widehat{T}(X,Y) = 2g(X, \varphi Y)\xi, \end{gathered} \tag{2.7}$$

respectively.

## 3 Slant curves in contact pseudo-Hermitian geometry

Let $M = \{M, \varphi, \xi, \eta, g\}$ be a contact metric 3-manifold and $\gamma : I \to M$ a curve parametrized by arc-length in $M$. The Frenet frame field $\{T, N, B\}$ along $\gamma$ for the pseudo-Hermitian connection $\widehat{\nabla}$ can be defined by

$$\begin{aligned} \widehat{\nabla}_T T &= \widehat{\kappa} N, \\ \widehat{\nabla}_T N &= -\widehat{\kappa} T + \widehat{\tau} B, \\ \widehat{\nabla}_T B &= -\widehat{\tau} N, \end{aligned} \tag{3.1}$$

where $\widehat{\kappa} = \left\| \widehat{\nabla}_T T \right\|$ is the *pseudo-Hermitian curvature* of $\gamma$ and $\widehat{\tau}$ its *pseudo-Hermitian torsion* [8]. Similar to the general curve theory, a curve, whose pseudo-Hermitian curvature and pseudo-Hermitian torsion are non-zero constants, is called a *pseudo-Hermitian helix.* Curves with constant non-zero pseudo-Hermitian curvature and zero pseudo-Hermitian torsion are called *pseudo-Hermitian circles*. *Pseudo-Hermitian geodesics* are curves whose pseudo-Hermitian curvature and pseudo-Hermitian torsion are zero [8].

Let $\gamma : I \to M$ be a Frenet curve parametrized by arc-length parameter $s$ in a contact metric 3-manifold $M$. The contact angle $\alpha(s)$ is a function defined by $cos[\alpha(s)] = g(T(s), \xi)$. If the contact angle $\alpha(s)$ is a constant, then $\gamma$ is called a *slant curve* [9]. Slant curves with contact angle $\pi/2$ are traditionally called *Legendre curves* [3].

Throughout the present paper, we assume that all curves are non-geodesic Frenet curves, that is, $\widehat{\kappa} \neq 0$.

In [8], Cho and Lee proved the following three propositions:

**Proposition 3.1.** [8] *A curve $\gamma$ for $\widehat{\nabla}$ is a slant curve if and only if it satisfies $\eta(N) = 0$.*

**Proposition 3.2.** [8] *Let $\gamma$ be a slant curve for $\widehat{\nabla}$ in a 3-dimensional contact metric manifold $M$. Then the ratio of $\widehat{\tau}$ and $\widehat{\kappa}$ is a constant.*

Note that

$$\frac{\widehat{\tau}}{\widehat{\kappa}} = \cot \alpha_0, \tag{3.2}$$

where $\alpha_0$ is the contact angle of $\gamma$ [14].

**Proposition 3.3.** [8] *If a curve in a 3-dimensional contact metric manifold for Tanaka-Webster connection $\widehat{\nabla}$ is a Legendre curve, then $\widehat{\tau} = 0$.*

In [14], the present author and Güvenç showed that the converse statement of the above proposition is also true. They gave the following result:

**Corollary 3.4.** [14] *Let $\gamma$ be a slant curve for Tanaka-Webster connection $\widehat{\nabla}$ with contact angle $\alpha_0$ in a 3-dimensional contact metric manifold $M$. Then $\gamma$ is a Legendre curve if and only if $\widehat{\tau} = 0$.*

## 4 Pseudo-Hermitian mean curvature vector field

The *pseudo-Hermitian mean curvature vector field* $\widehat{H}$ of a curve $\gamma$ in a 3-dimensional contact metric manifold is defined by

$$\widehat{H} = \widehat{\nabla}_T T = \widehat{\kappa} N, \tag{4.1}$$

(see [12]). From (4.1), it is easy to see that

$$\widehat{\nabla}_T \widehat{H} = -\widehat{\kappa}^2 T + \widehat{\kappa}' N + \widehat{\kappa}\widehat{\tau} B, \tag{4.2}$$

$$\widehat{\nabla}^{\perp}_{T}\widehat{H} = \widehat{\kappa}'N + \widehat{\kappa}\widehat{\tau}B, \tag{4.3}$$

where $\widehat{H}$ is the pseudo-Hermitian mean curvature vector field of $\gamma$ [12].

**Definition 4.1.** Let $H$ be the mean curvature vector field of a curve $\gamma$ in a 3-dimensional contact metric manifold $M$. The mean curvature vector field $H$ is said to be *pseudo-Hermitian $C$-parallel* if $\widehat{\nabla}_T\widehat{H} = \lambda\xi$. The vector field $H$ is said to be *pseudo Hermitian $C$-proper mean curvature vector field* if $\widehat{\triangle}\widehat{H} = \lambda\xi$. Similarly, $H$ is said to be *pseudo-Hermitian $C$-parallel vector field in the normal bundle* if $\widehat{\nabla}^{\perp}_{T}\widehat{H} = \lambda\xi$, and $H$ is said to be *pseudo-Hermitian $C$-proper mean curvature vector field in the normal bundle* if $\widehat{\triangle}^{\perp}\widehat{H} = \lambda\xi$, where $\lambda$ is a differentiable function along the curve.

**Lemma 4.2.** *[14]Let $\gamma$ be a curve in a 3-dimensional contact metric manifold $M$. Then*

$$\widehat{\nabla}_T\widehat{\nabla}_T\widehat{\nabla}_T T = -3\widehat{\kappa}\widehat{\kappa}'T + (\widehat{\kappa}'' - \widehat{\kappa}^3 - \widehat{\kappa}\widehat{\tau}^2)N + (2\widehat{\kappa}'\widehat{\tau} + \widehat{\kappa}\widehat{\tau}')B, \tag{4.4}$$

$$\widehat{\nabla}^{\perp}_{T}\widehat{\nabla}^{\perp}_{T}\widehat{\nabla}^{\perp}_{T}T = (\widehat{\kappa}'' - \widehat{\kappa}\widehat{\tau}^2)N + (2\widehat{\kappa}'\widehat{\tau} + \widehat{\kappa}\widehat{\tau}')B \tag{4.5}$$

*and*

$$\begin{aligned} \widehat{\triangle}\widehat{H} &= -\widehat{\nabla}_T\widehat{\nabla}_T\widehat{\nabla}_T T, \\ \widehat{\triangle}^{\perp}\widehat{H} &= -\widehat{\nabla}^{\perp}_{T}\widehat{\nabla}^{\perp}_{T}\widehat{\nabla}^{\perp}_{T}T. \end{aligned} \tag{4.6}$$

Using Lemma 4.2, we have the following theorem:

**Theorem 4.3.** *A slant curve $\gamma$ in a 3-dimensional contact metric manifold $M$ has pseudo-Hermitian $C$-parallel mean curvature vector field if and only if it is a pseudo-Hermitian helix satisfying*

$$\widehat{\kappa} = \mp\sqrt{-\lambda\cos\alpha_0} \quad \text{and} \quad \widehat{\tau} = \mp\frac{\lambda\sin\alpha_0}{\sqrt{-\lambda\cos\alpha_0}},$$

*where $\lambda\cos\alpha_0 < 0$.*

*Proof.* Assume that a slant curve $\gamma$ has pseudo-Hermitian $C$-parallel mean curvature vector field. Then from (4.2), the condition $\widehat{\nabla}_T\widehat{H} = \lambda\xi$ gives

$$-\widehat{\kappa}^2T + \widehat{\kappa}'N + \widehat{\kappa}\widehat{\tau}B = \lambda\xi. \tag{4.7}$$

Since $\gamma$ is a slant curve we can write

$$\xi = \cos\alpha_0 T + \sin\alpha_0 B. \tag{4.8}$$

So using (4.7) and (4.8) we can write

$$-\widehat{\kappa}^2T + \widehat{\kappa}'N + \widehat{\kappa}\widehat{\tau}B = \lambda\left(\cos\alpha_0 T + \sin\alpha_0 B\right). \tag{4.9}$$

Taking the inner product of (4.9) with $N$ and using $\eta(N) = 0$ we find $\widehat{\kappa}' = 0$, which implies that $\widehat{\kappa}$ is a constant. Hence from the equation (4.9), it follows that $\widehat{\kappa} = \mp\sqrt{-\lambda\cos\alpha_0}$. Since $\frac{\widehat{\tau}}{\widehat{\kappa}} = \cot\alpha_0$ we obtain $\widehat{\tau} = \mp\frac{\lambda\sin\alpha_0}{\sqrt{-\lambda\cos\alpha_0}}$, where $\lambda\cos\alpha_0 < 0$.

The converse statement is trivial. □

**Theorem 4.4.** *A slant curve $\gamma$ in a 3-dimensional contact metric manifold $M$ has pseudo-Hermitian $C$-parallel mean curvature vector field in the normal bundle if and only if it is a pseudo-Hermitian Legendre circle.*

*Proof.* Assume that a slant curve $\gamma$ has pseudo-Hermitian $C$-parallel mean curvature vector field in the normal bundle. Then from (4.2), the condition $\widehat{\nabla}^{\perp}_{T}\widehat{H} = \lambda\xi$ gives

$$\widehat{\kappa}' N + \widehat{\kappa}\widehat{\tau} B = \lambda\left(\cos\alpha_0 T + \sin\alpha_0 B\right). \tag{4.10}$$

So we have

$$\widehat{\kappa}' = 0, \tag{4.11}$$

$$\lambda\cos\alpha_0 = 0, \tag{4.12}$$

$$\widehat{\kappa}\widehat{\tau} = \lambda\sin\alpha_0. \tag{4.13}$$

Then $\widehat{\kappa}$ is a constant. From (4.12), if $\cos\alpha_0 = 0$, then $\alpha_0 = \pi/2$. So it is a Legendre curve. Then from Proposition 3.4, $\widehat{\tau} = 0$, which implies $\gamma$ is a pseudo-Hermitian Legendre circle. Moreover, from (4.13) we have $\lambda = 0$.

The converse statement is trivial. □

**Theorem 4.5.** *There does not exist non-geodesic slant curve in a 3-dimensional contact metric manifold $M$ with pseudo Hermitian $C$-proper mean curvature.*

*Proof.* Assume that $\gamma$ is a non-geodesic slant curve with contact angle $\alpha_0$ and has pseudo Hermitian $C$-proper mean curvature field. Then by definition, $\widehat{\triangle}\widehat{H} = \lambda\xi$. Using (4.6) and (4.8), we get

$$\begin{aligned}3\widehat{\kappa}\widehat{\kappa}' T - (\widehat{\kappa}'' - \widehat{\kappa}^3 - \widehat{\kappa}\widehat{\tau}^2)N - (2\widehat{\kappa}'\widehat{\tau} + \widehat{\kappa}\widehat{\tau}')B \\ = \lambda\left(\cos\alpha_0 T + \sin\alpha_0 B\right).\end{aligned} \tag{4.14}$$

Hence we have

$$3\widehat{\kappa}\widehat{\kappa}' = \lambda\cos\alpha_0,$$

$$\widehat{\kappa}'' - \widehat{\kappa}^3 - \widehat{\kappa}\widehat{\tau}^2 = 0,$$

$$2\widehat{\kappa}'\widehat{\tau} + \widehat{\kappa}\widehat{\tau}' = -\lambda\sin\alpha_0.$$

So using $\frac{\widehat{\tau}}{\widehat{\kappa}} = \cot\alpha_0$, we find $\lambda = 0$. Then using Theorem 4.4. in [14], we find $\widehat{\kappa} = 0$. Since $\gamma$ is not a geodesic, it can not have pseudo Hermitian $C$-proper mean curvature.

This completes the proof. □

**Theorem 4.6.** *A slant curve $\gamma$ in a 3-dimensional contact metric manifold $M$ has pseudo-Hermitian $C$-proper mean curvature field in the normal bundle if and only if either it is a Legendre curve with pseudo-Hermitian curvature $\widehat{\kappa}(s) = as + b$, where $a$ and $b$ are real constants or it is a pseudo-Hermitian Legendre circle.*

*Proof.* Assume that $\gamma$ is a non-geodesic slant curve with contact angle $\alpha_0$ and has pseudo Hermitian $C$-proper mean curvature vector field in the normal bundle. Then by definition, $\widehat{\triangle}^{\perp}\widehat{H} = \lambda\xi$. Using (4.6) and (4.8), we get

$$-(\widehat{\kappa}'' - \widehat{\kappa}\widehat{\tau}^2)N - (2\widehat{\kappa}'\widehat{\tau} + \widehat{\kappa}\widehat{\tau}')B = \lambda\left(\cos\alpha_0 T + \sin\alpha_0 B\right).$$

Then we have

$$\widehat{\kappa}'' - \widehat{\kappa}\widehat{\tau}^2 = 0, \tag{4.15}$$

$$-(2\widehat{\kappa}'\widehat{\tau} + \widehat{\kappa}\widehat{\tau}') = \lambda\sin\alpha_0, \tag{4.16}$$

$$\lambda\cos\alpha_0 = 0. \tag{4.17}$$

From (4.17), if $\cos\alpha_0 = 0$, then $\alpha_0 = \pi/2$. So it is a Legendre curve. Then from Proposition 3.4, $\widehat{\tau} = 0$. Thus the equations (4.15) and (4.16) give us $\lambda = 0$. Then by Theorem 4.7 in [14], it follows that $\gamma$ is a Legendre curve with pseudo-Hermitian curvature $\widehat{\kappa}(s) = as + b$, where $a$ and $b$ are real constants. If $\cos\alpha_0 \neq 0$ and $\lambda = 0$ then in view of Theorem 4.7 in [14], it follows that $\gamma$ is a pseudo-Hermitian Legendre circle.

The converse statement is trivial. □

# 5 Slant curves of Sasakian Heisenberg Group with Pseudo-Hermitian Connection

The Heisenberg group $H_3$ can be viewed as $\mathbb{R}^3$ equipped with Riemannian metric

$$g = dx^2 + dy^2 + \eta \otimes \eta,$$

where $(x, y, z)$ are standard coordinates in $\mathbb{R}^3$ and

$$\eta = dz + ydx - xdy.$$

The 1-form $\eta$ satisfies $d\eta \wedge \eta = -\lambda dx \wedge dy \wedge dz$. Hence $\eta$ is a contact form. In [10], Inoguchi obtained the Levi-Civita connection $\nabla$ of the metric $g$ with respect to the left-invariant orthonormal basis

$$e_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z},\ e_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z},\ e_3 = \frac{\partial}{\partial z}.$$

He obtained

$$\begin{array}{lll} \nabla_{e_1}e_1 = 0, & \nabla_{e_1}e_2 = e_3, & \nabla_{e_1}e_3 = -e_2, \\ \nabla_{e_2}e_1 = -e_3, & \nabla_{e_2}e_2 = 0, & \nabla_{e_2}e_3 = e_1, \\ \nabla_{e_3}e_1 = -e_2, & \nabla_{e_3}e_2 = e_1, & \nabla_{e_3}e_3 = 0. \end{array} \tag{5.1}$$

We also have the Heisenberg brackets

$$[e_1, e_2] = 2e_3, \quad [e_2, e_3] = [e_3, e_1] = 0.$$

Let $\varphi$ be the $(1,1)$-tensor field defined by $\varphi(e_1) = e_2$, $\varphi(e_2) = -e_1$ and $\varphi(e_3) = 0$. Then using the linearity of $\varphi$ and $g$ we have

$$\eta(e_3) = 1, \quad \varphi^2(X) = -X + \eta(X)e_3, \quad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y).$$

We also have

$$d\eta(X,Y) = g(X, \varphi Y)$$

for all $X, Y \in \chi(M)$. Thus for $\xi = e_3$, $(\varphi, \xi, \eta, g)$ is a contact metric structure and the Heisenberg group $H_3$ is a Sasakian space form of constant holomorphic sectional curvature $-3$ [10].

Now, let $\gamma : I \to H_3$ be a slant curve with contact angle $\alpha_0$. Assume that $\gamma$ is parametrized by arc length $s$ and $\{T, N, B\}$ denote the Frenet frame of $\gamma$. Then we can write

$$T = \sin\alpha_0 \cos\beta e_1 + \sin\alpha_0 \sin\beta e_2 + \cos\alpha_0 e_3, \tag{5.2}$$

where $\beta = \beta(s)$. Using (5.1) we have

$$\begin{aligned} \nabla_T T &= \left(-\sin\alpha_0 \sin\beta \left(\beta' - 2\cos\alpha_0\right)\right) e_1 \\ &\quad + \left(\sin\alpha_0 \cos\beta \left(\beta' - 2\cos\alpha_0\right)\right) e_2. \end{aligned} \tag{5.3}$$

On the other hand by the use of (5.2) it follows that

$$\varphi T = -\sin\alpha_0 \sin\beta e_1 + \sin\alpha_0 \cos\beta e_2. \tag{5.4}$$

By the use of (2.7) we find

$$\widehat{\nabla}_T T = -\beta' \sin\alpha_0 \sin\beta e_1 + \beta' \sin\alpha_0 \cos\beta e_2. \tag{5.5}$$

Since $\widehat{\nabla}_T T = \widehat{\kappa} N$, the equation (5.5) gives us

$$\widehat{\kappa} = |\beta'| \sin\alpha_0. \tag{5.6}$$

Hence the principal normal vector field $N$ of $\gamma$ can be written as

$$N = sgn(\beta')\left(-\sin\beta e_1 + \cos\beta e_2\right).$$

Since $B = T \times N$, we find

$$B = sgn(\beta')\left(-\cos\alpha_0 \cos\beta e_1 - \cos\alpha_0 \sin\beta e_2 + \sin\alpha_0 e_3\right).$$

Then it is easy to see that

$$B' = sgn(\beta')\left(\beta' \cos\alpha_0 - \cos 2\alpha_0\right)\left(\sin\beta e_1 - \cos\beta e_2\right),$$

which gives us

$$\widehat{\tau} = \beta' \cos\alpha_0 - \cos 2\alpha_0. \tag{5.7}$$

Now assume that $\widehat{\tau}=0$. Then from Proposition 3.4, $\gamma$ is a Legendre curve. Using (5.7), we obtain

$$\beta(s)=\frac{\cos 2\alpha_0}{\cos\alpha_0}s+c,$$

where $c$ is a real constant. Hence from (5.6), $\widehat{\kappa}$ is a constant.

Let $\gamma(s)=(x(s),y(s),z(s))$. To find the explicit equations, we should integrate the system $\frac{d\gamma}{ds}=T$. Then

$$\frac{dx}{ds}=\sin\alpha_0\cos\left(\frac{\cos 2\alpha_0}{\cos\alpha_0}s+c\right),$$

$$\frac{dy}{ds}=\sin\alpha_0\sin\left(\frac{\cos 2\alpha_0}{\cos\alpha_0}s+c\right),$$

$$\frac{dz}{ds}=\cos\alpha_0+\frac{1}{2}\sin\alpha_0\left(\sin\left(\frac{\cos 2\alpha_0}{\cos\alpha_0}s+c\right)x(s)-\cos\left(\frac{\cos 2\alpha_0}{\cos\alpha_0}s+c\right)y(s)\right).$$

So using the method given in [4], the integration of above system gives the following example:

*Example* 5.1. Let $\gamma: I\to H_3$ be a curve with the following parametric equations.

$$x(s)=\frac{\cos\alpha_0}{\cos 2\alpha_0}\sin\alpha_0\sin(\frac{\cos 2\alpha_0}{\cos\alpha_0}s+c)+d_1,$$

$$y(s)=-\frac{\cos\alpha_0}{\cos 2\alpha_0}\sin\alpha_0\cos(\frac{\cos 2\alpha_0}{\cos\alpha_0}s+c)+d_2,$$

$$\begin{aligned} z(s) &= \left(\cos\alpha_0+\frac{\cos\alpha_0}{\cos 2\alpha_0}\sin^2\alpha_0\right)s-d_1\frac{\cos\alpha_0}{\cos 2\alpha_0}\sin\alpha_0\cos(\frac{\cos 2\alpha_0}{\cos\alpha_0}s+c)\\ &\quad -d_2\frac{\cos\alpha_0}{\cos 2\alpha_0}\sin\alpha_0\sin(\frac{\cos 2\alpha_0}{\cos\alpha_0}s+c)+d_3.\end{aligned}$$

Then $\gamma$ is a pseudo-Hermitian Legendre circle with *pseudo-Hermitian curvature* $\widehat{\kappa}=\left|\frac{\cos 2\alpha_0}{\cos\alpha_0}\right|\sin\alpha_0$, where $c,d_1,d_2$ and $d_3$ are some real constants.

Now assume that $\widehat{\tau}\neq 0$ and $\widehat{\kappa}$ is a constant. Then from (5.6), $\beta'$ is a constant. Then we can write $\beta(s)=as+b$, where $a$ and $b$ are real constants. By the use of equation (5.7), we find $\widehat{\tau}=a\cos\alpha_0-\cos 2\alpha_0$. Hence $\widehat{\tau}$ is a constant. Similar to the method using in previous example, let $\gamma(s)=(x(s),y(s),z(s))$. To find the explicit equations, we should integrate the system $\frac{d\gamma}{ds}=T$. Then

$$\frac{dx}{ds}=\sin\alpha_0\cos(as+b),$$

$$\frac{dy}{ds}=\sin\alpha_0\sin(as+b),$$

$$\frac{dz}{ds}=\cos\alpha_0+\frac{1}{2}\sin\alpha_0\left(\sin(as+b)\,x(s)-\cos(as+b)\,y(s)\right).$$

Similarly, using the method given in [4], the integration of above system gives the following example:

*Example* 5.2. Let $\gamma : I \to H_3$ be a curve with the following parametric equations.

$$x(s) = \frac{1}{a}\sin\alpha_0 \sin(as+b) + c_1,$$

$$y(s) = -\frac{1}{a}\sin\alpha_0 \cos(as+b) + c_2,$$

$$\begin{aligned} z(s) &= \left(\cos\alpha_0 + \frac{1}{a}\sin^2\alpha_0\right)s - \frac{c_1}{a}\sin\alpha_0\cos(as+b) \\ &\quad -\frac{c_2}{a}\sin\alpha_0\sin(as+b) + c_3. \end{aligned}$$

Then $\gamma$ is a pseudo-Hermitian slant helix with *pseudo-Hermitian curvature* $\widehat{\kappa} = |a|\sin\alpha_0$ and *pseudo-Hermitian torsion* $\widehat{\tau} = a\cos\alpha_0 - \cos 2\alpha_0$ , where $a, b, c_1, c_2$ and $c_3$ are some real constants such that $a \neq \frac{\cos 2\alpha_0}{\cos\alpha_0}$.

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