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A Survey on Submanifolds with Nonpositive Extrinsic Curvature

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Abstract. We survey on some recent developments on the study of submanifolds with nonpositive extrinsic curvature.
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1 Introduction

One of the main problems in submanifold theory is to know whether given complete Riemannian manifolds M^m and N^n, with $m < n$, there exists an isometric immersion $f : M^m \to N^n$. In case the ambient space is the Euclidean space, the Nash embedding theorem says that there is an isometric embedding $f : M^m \to \mathbb{R}^n$ provided the codimension $n - m$ is sufficiently large. For small codimension, the answer in general depends on the geometries of M and N. Isometric immersions $f : M^m \to N^n$ with low codimension and nonpositive extrinsic curvature at any point must satisfy strong geometric conditions. The simplest result along this line is that a surface with nonpositive curvature in $\mathbb{R}^3$ cannot be compact. This is a consequence of the well-know fact that at a point of maximum of a distance function on a compact surface in $\mathbb{R}^3$ the Gaussian curvature must be positive.

In the same direction, the Hilbert-Efimov theorem [4], [5] states that no complete surface M with sectional curvature $K_M \leq -\delta^2 < 0$ can be isometrically immersed in $\mathbb{R}^3$. A classical result by Tompkins [17] states that a compact flat m-dimensional Riemannian manifold cannot be isometrically immersed in $\mathbb{R}^{2m-1}$. Tompkins's result was extended in a series of papers by Chern and Kuiper [3], Moore [8], O'Neill [11], Otsuki [12] and Stiel [15], whose results can be summarized as follows:

Theorem 1.1. *Let $f : M^m \to N^n$ be an isometric immersion of a compact Riemannian manifold M into a Cartan-Hadamard manifold N, with $n \leq 2m-1$. Then the sectional curvatures of M and N satisfy*

$$\sup_M K_M > \inf_N K_N .$$

The aim of this paper is to survey on some recent extensions of Theorem 1.1, mostly for the case of complete cylindrically bounded submanifolds.

2 Bounded complete submanifolds with scalar curvature bounded from below

Let $f : M^m \to N^n$ be an isometric immersion. In the statement below and the sequel, ρ stands for the distance function to a given reference point in M^m, $\log^{(j)}$ is the j-th iterate of the logarithm and $t \gg 1$ means that t is sufficiently large. Also $B_N[R]$ denotes the closed geodesic ball with radius $0 < R < \min\left\{\mathrm{inj}_N(o), \pi/2\sqrt{b}\right\}$ centered at a point o of N^n, where $\mathrm{inj}_N(o)$ is the injectivity radius of N^n at o and $\pi/2\sqrt{b}$ is replaced by $+\infty$ if $b \leq 0$. Moreover, $K_M(\sigma)$ denotes the sectional curvature of M^m at a point $x \in M^m$ along the plane $\sigma \subset T_x M$, and similarly for N^n,

$$K_f(\sigma) := K_M(\sigma) - K_N(f_* \sigma)$$

is the *extrinsic sectional curvature* of f at x along σ and K_N^{rad} stands for the radial sectional curvature of N^n with respect to o, that is, the sectional curvature of tangent planes to N^n containing the vector $\mathrm{grad}^N r$, where r is the distance function to o in N^n. Finally, let C_b be the real function given by

$$C_b(t) = \begin{cases} \sqrt{b}\cot(\sqrt{b}t) & \text{if } b > 0 \text{ and } 0 < t < \dfrac{\pi}{2\sqrt{b}}, \\ \frac{1}{t} & \text{if } b = 0 \text{ and } t > 0, \\ \sqrt{-b}\coth(\sqrt{-b}t) & \text{if } b < 0 \text{ and } t > 0. \end{cases}$$

Theorem 1.1 was extended by Jorge and Koutrofiotis [6] to bounded complete submanifolds with scalar curvature bounded from below. Pigola, Rigoli and Setti presented in [13] an extension of Theorem 1.1 with scalar curvature satisfying

$$s_M(x) \geq -A^2 \rho^2(x) \prod_{j=1}^{J} \left(\log^{(j)}(\rho(x))\right)^2, \ \rho(x) \gg 1, \tag{2.1}$$

for some constant $A > 0$ and some integer $J \geq 1$ (where we use the definition in which the scalar curvature and also the Ricci curvature in Section 5 are divided by $m-1$).

Theorem 2.1 ([13])**.** *Let $f : M^m \to N^n$ be an isometric immersion with codimension $p = n - m < m$ of a complete Riemannian manifold whose scalar curvature satisfies* (2.1)*. Assume that $f(M) \subset B_N[R]$. If $K_N^{\mathrm{rad}} \le b$ in $B_N[R]$, then*

$$\sup_M K_M \ge C_b^2(R) + \inf_{B_N[R]} K_N. \tag{2.2}$$

Note that if $N^n = \mathbb{Q}_b^n$ is the simply connected space form of constant sectional curvature b and $M = \partial B_{\mathbb{Q}_b^n}[R] \subset \mathbb{Q}_b^n$ is a geodesic sphere of radius R, then equality (2.2) is achieved.

3 Cylindrically bounded submanifolds

In this section we will discuss an extension of Theorem 2.1 due to Alías, Bessa and Montenegro for the case of cylindrically bounded submanifolds. More precisely, in [1] they have provided an estimate for the extrinsic curvatures of complete cylindrically bounded submanifolds of a Riemannian product $P^n \times \mathbb{R}^k$, where *cylindrically bounded* means that there exists a (closed) geodesic ball $B_P[R]$ of P^n, centered at a point $o \in P^n$ with radius satisfying $0 < R < \min\left\{\mathrm{inj}_P(o), \pi/2\sqrt{b}\right\}$ (where $\pi/2\sqrt{b}$ is replaced by $+\infty$ if $b \le 0$), such that

$$f(M) \subset B_P[R] \times \mathbb{R}^k. \tag{3.1}$$

Otherwise, we say that f is *cylindrically unbounded.*

Theorem 3.1 ([1])**.** *Let $f : M^m \to P^n \times \mathbb{R}^k$ be an isometric immersion with codimension $p = n + k - m < m - k$ of a complete Riemannian manifold whose scalar curvature satisfies* (2.1)*. Assume that f is cylindrically bounded and that P^n is complete. If $K_P^{\mathrm{rad}} \le b$ in $B_P[R]$, then*

$$\sup_M K_f \ge C_b^2(R). \tag{3.2}$$

Moreover,

$$\sup_M K_M \ge C_b^2(R) + \inf_{B_P[R]} K_P. \tag{3.3}$$

We point out that the codimension restriction $p < m - k$ cannot be relaxed. Actually, it implies that $n > 2$ and $m > k + 1$. In particular, in a three-dimensional ambient space N^3, that is, $n + k = 3$, we have that $k = 0$, and therefore $f(M) \subset B_P[R]$. In fact, the flat cylinder $\mathbb{S}^1(R) \times \mathbb{R} \subset B_{\mathbb{R}^2}[R] \times \mathbb{R}$ shows that the restriction $p < m - k$ is necessary.

On the other hand, estimates (3.2) and (3.3) are sharp. Indeed, the function C_b is well-known: the geodesic sphere $\partial B_{\mathbb{Q}_b^m}(R)$ of radius R in the simply connected complete space form $\mathbb{Q}_b^m$ of constant sectional curvature b, with $R < \frac{\pi}{2\sqrt{b}}$ if $b > 0$, is an umbilical hypersurface with principal curvatures being precisely $C_b(R)$. This shows that its extrinsic and intrinsic sectional curvatures are constant and equal to $C_b^2(R)$ and $C_b^2(R) + b$, respectively, the latter following from the former by the Gauss equation. Then, for every $m > 2$ and $k \geq 0$ we can consider $M^{m-1+k} = \partial B_{\mathbb{Q}_b^m}(R) \times \mathbb{R}^k$ and take $f : M^{m-1+k} \to B_{\mathbb{Q}_b^m}[R] \times \mathbb{R}^k$ to be the canonical isometric embedding. Therefore $\sup_M K_f$ and $\sup_M K_M$ are the constant extrinsic and intrinsic sectional curvatures $C_b^2(R)$ and $C_b^2(R) + b$ of $\partial B_{\mathbb{Q}_b^m}(R)$, respectively.

Remark 3.2. The geometry of the Euclidean factor $\mathbb{R}^k$ plays essentially no role in the proof of Theorem 3.1. Indeed, estimate (3.3) remains true if the former is replaced by any Riemannian manifold Q^k, which need not be even complete, whereas for (3.2) the only requirement is that K_Q be bounded from above. In the next section we will discuss a more accurate conclusion than the one of Theorem 3.1 (see Theorem 4.1 and comment below).

As a consequence of Theorem 3.1, the following results about extrinsic radius were obtained.

Theorem 3.3. *[1] Let $f : M^m \to P^n \times \mathbb{R}^l$ be an isometric immersion of a compact Riemannian M^m with codimension $p = n + l - m < m - l$. Assume that P^n is a complete Riemannian manifolds with a pole and radial sectional curvature $K_P^{rad} \leq b \leq 0$. Then, the extrinsic radius satisfies*

$$R_f \geq C_b^{-1}\left(\sqrt{\sup K_M - \inf K_N}\right).$$

In particular, if $P^n = \mathbb{R}^n$ we have that

$$R_f \geq \frac{1}{\sqrt{\sup K_M}}.$$

Theorem 3.4. *[1] Let $f : M^m \to \mathbb{S}^n \times \mathbb{R}^l$ be an isometric immersion of a compact Riemannian M^m with codimension $p = n+l-m < m-l$. If $\sup K_M \leq 1$, then*

$$R_f \geq \frac{\pi}{2}.$$

4 Cylindrically bounded submanifolds: a more general setting

The purpose of this section is to discuss a more accurate conclusion than the one of Theorem 3.1. More precisely, the authors [2] understood how much extrinsic (respectively, intrinsic) sectional curvature satisfying estimate (3.2) (respectively (3.3)) appears depending on how low the codimension is. The idea is that the lower the codimension is, the more extrinsic (respectively, intrinsic) sectional curvature satisfying (3.2) (respectively (3.3)) will appear.

In the same way as in (3.1), an isometric immersion $f : M^m \to P^n \times Q^k$ is said to be *cylindrically bounded* if there exists a (closed) geodesic ball $B_P[R]$ of P^n, centered at a point $o \in P^n$ with radius $R > 0$, such that

$$f(M) \subset B_P[R] \times Q^k, \tag{4.1}$$

with $0 < R < \min\left\{\operatorname{inj}_P(o), \frac{\pi}{2\sqrt{b}}\right\}$, where $\frac{\pi}{2\sqrt{b}}$ is replaced by $+\infty$ if $b \le 0$.

Theorem 4.1 ([2])**.** *Let $f : M^m \to P^n \times Q^k$ be an isometric immersion with codimension $p = n + k - m < m - k$ of a complete Riemannian manifold whose radial sectional curvature $K_M^{\mathrm{rad}}(x)$ satisfies*

$$K_M^{\mathrm{rad}}(x) \ge -A^2 \rho^2(x) \prod_{j=1}^{J} \left(\log^{(j)}(\rho(x))\right)^2, \; \rho(x) \gg 1. \tag{4.2}$$

Assume that f is cylindrically bounded. If $K_P^{\mathrm{rad}} \le b$ in $B_P[R]$, then

$$\sup_M \min\left\{\max_{\sigma \subset W} K_f(\sigma) : \dim W > p + k\right\} \ge C_b^2(R). \tag{4.3}$$

Moreover,

$$\sup_M \min\left\{\max_{\sigma \subset W} K_M(\sigma) : \dim W > p + k\right\} \ge C_b^2(R) + \inf_{B_P[R]} K_P. \tag{4.4}$$

The estimates of Theorem 4.1 are clearly better than the ones of Theorem 3.1. Actually, (4.3) and (4.4) reduce to (3.2) and (3.3), respectively, only in the case of the highest allowed codimension $p = m - 1 - k$. On the other hand, although one has a stronger assumption on the curvature of M^m, if (2.1) holds but (4.2) does not, then, since the scalar curvature is an average of sectional curvatures, we have that $\sup_M K_M = +\infty$, and hence (3.3) is trivially satisfied. Moreover, K_P is clearly bounded in $B_P[R]$, thus if also K_Q is bounded from above, we conclude that $\sup_M K_f = +\infty$ by the Gauss equation, so that (3.2) also holds trivially in this case. Finally, note that the same example considered below Theorem 3.1 shows that our estimates (4.3) and (4.4) are also sharp.

5 Applications

In this section we will discuss some applications of Theorem 4.1. Denote by R_f the *extrinsic radius* of a cylindrically bounded isometric immersion f, that is, the smallest R for which (4.1) holds. A first application of Theorem 4.1 are the following versions of Theorem 3.3 and 3.4.

Corollary 5.1 ([2])**.** *Let $f : M^m \to P^n \times Q^k$ be an isometric immersion with codimension $p = n + k - m < m - k$ of a complete Riemannian manifold whose radial sectional curvature satisfies* (4.2)*. Assume that P^n is a complete Riemannian manifold with a pole and radial sectional curvatures $K_P^{\mathrm{rad}} \le b \le 0$. If f is cylindrically bounded, then*

$$\sup_M \min\left\{\max_{\sigma \subset W} K_f(\sigma) : \dim W > p + k\right\} > -b$$

and the extrinsic radius satisfies

$$R_f \geq C_b^{-1}\left(\sqrt{\sup_M \min\left\{\max_{\sigma\subset W} K_f\left(\sigma\right) : \dim W > p+k\right\}}\right). \tag{5.1}$$

In particular, if

$$\sup_M \min\left\{\max_{\sigma\subset W} K_f\left(\sigma\right) : \dim W > p+k\right\} \leq -b,$$

then f is cylindrically unbounded.

Corollary 5.2 ([2])**.** *Let $f : M^m \to \mathbb{S}^n \times Q^k$ be an isometric immersion with codimension $p = n+k-m < m-k$ of a complete Riemannian manifold whose radial sectional curvature satisfies* (4.2)*. If*

$$\sup_M \min\left\{\max_{\sigma\subset W} K_M\left(\sigma\right) : \dim W > p+k\right\} \leq 1,$$

then

$$R_f \geq \frac{\pi}{2}. \tag{5.2}$$

On the other hand, a sharp lower bound for the Ricci curvature of bounded complete Euclidean hypersurfaces was obtained by Leung [7] and extended by Veeravalli [18] to nonflat ambient space forms. For simplicity of notation we shall denote by $\sup_M Ric(M)$ the $\sup_{X\in UM} Ric(X,X)$, where UM is the unitary tangent bundle.

Theorem 5.3 ([18])**.** *Let $f : M^m \to \mathbb{Q}_b^{m+1}$ be a complete hypersurface with sectional curvature bounded away from $-\infty$ such that $f(M) \subset B_{\mathbb{Q}_b^{m+1}}[R]$, with $R < \frac{\pi}{2\sqrt{b}}$ if $b > 0$. Then*

$$\sup_M \mathrm{Ric}(M) \geq C_b^2\left(R\right) + b. \tag{5.3}$$

Theorem 4.1 also gives an improvement of the above result, where we consider hypersurfaces of much more general ambient spaces and obtain that estimate (5.3) actually holds for the scalar curvature. This shows the unifying character of Theorem 4.1.

Corollary 5.4 ([2])**.** *Let $f : M^m \to P^{m+1}$ be a complete hypersurface whose radial sectional curvatures satisfy* (4.2)*. Assume that $f(M) \subset B_P[R]$, with R as in Theorem 4.1. If $K_P^{\mathrm{rad}} \leq b$ in $B_P[R]$, then*

$$\sup_M s_M \geq C_b^2\left(R\right) + \inf_{B_P[R]} K_P.$$

Again observe that for the geodesic sphere $M^m = \partial B_{\mathbb{Q}_b^{m+1}}(R)$ of radius R in $\mathbb{Q}_b^{m+1}$ the above inequality is in fact an equality. Corollary 5.5 leads to similar extrinsic radius results to Corollaries 5.1 and 5.2 and, in particular, a criterion of unboundedness:

Corollary 5.5 ([2])**.** *Let $f : M^m \to P^{m+1}$ be a complete hypersurface whose radial sectional curvatures satisfy* (4.2)*. Assume that P^{m+1} is a complete Riemannian manifold with a pole and sectional curvatures $K_P \geq c$ and $K_P^{\text{rad}} \leq b \leq 0$. If $f(M)$ is bounded, then $\sup_M s_M > c - b$ and*

$$R_f \geq C_b^{-1}\left(\sqrt{\sup_M s_M - c}\right).$$

In particular, if $\sup_M s_M \leq c - b$, then $f\,(M)$ is unbounded.

Corollary 5.6 ([2])**.** *Let $f : M^m \to \mathbb{S}^{m+1}$ be a complete hypersurface whose radial sectional curvature satisfies* (4.2)*. If $\sup_M s_M \leq 1$, then*

$$R_f \geq \frac{\pi}{2}.$$

Remark 5.7. One of the main tools to prove this kind of result, in particular Theorem 4.1, is an algebraic lemma due to Otsuki [12], about symmetric bilinear forms. On the other hand, a key ingredient to handle the noncompact case is a maximum principle due to Omori [10] and generalized by Pigola-Rigoli-Setti [13].

6 Conjecture

One of the most important open problems in the area of geometry of submanifolds is an old conjecture on the higher-dimensional extension of Hilbert's classical theorem asserting that the complete hyperbolic plane $\mathbb{H}^2$ cannot be isometrically immersed into three-dimensional Euclidean space $\mathbb{R}^3$. Hilbert's theorem was proven at the turn of the last century in [5] and was one of the first *global* theorems from the Riemannian geometry of surfaces. It is quite natural to explore whether this result could be extended to higher dimensions. It follows from Otsuki's lemma that there are no m-dimensional submanifolds of constant negative curvature in $\mathbb{R}^{2m-2}$. In $\mathbb{R}^{2m-1}$, Moore [9] showed that the existence of an isometric immersion $f : \mathbb{H}^m \to \mathbb{R}^{2m-1}$ implies the existence of a Chebyshev net on $\mathbb{H}^m$, thereby extending the main step in the standard proof of Hilbert's theorem to m dimensions. However, despite the effort of many geometers such as Tenenblat and Terng [16], Xavier [19], and Aminov, it is remarkable that the conjectured extension of Hilbert's theorem has not been solved yet even in the next case $m = 3$. Most of the attempts were made by trying to face the problem directly, exploring the fairly complete understanding of the structure of m-dimensional submanifolds of constant curvature in $\mathbb{R}^{2m-1}$ provided by the study of the fundamental equations to reduce the question to a problem of global analysis generalizing the sine-Gordon equation. But as it often happens in mathematics, the answer for a conjecture may arise out of the solution of a more general problem. Hilbert's own theorem illustrates this point, since it is just the special constant curvature case of Efimov's much stronger statement that a complete surface with sectional curvature $K \leq -c < 0$ cannot be immersed isometrically in $\mathbb{R}^3$. Generalizations to higher dimensions of this stronger result have been in the direction of hypersurfaces [14] rather than to codimension $m-1$. Nevertheless, we point out that Theorem 4.1 leads to a conjecture that goes right into the latter direction. Indeed, it is a natural question to ask whether Theorem 4.1 is still true in the limiting case, that is, when $R = \mathrm{inj}_P(o) = \frac{\pi}{2\sqrt{b}}$, where $\frac{\pi}{2\sqrt{b}}$ is replaced by $+\infty$ if $b \leq 0$, which motivates the following:

Conjecture 6.1. *Let $f : M^m \to N^{n+l} = P^n \times Q^l$ be an isometric immersion with codimension $p = n + l - m < m - l$ of a complete Riemannian manifold. Assume that $R = \mathrm{inj}_P(o) = \frac{\pi}{2\sqrt{b}}$, where $\frac{\pi}{2\sqrt{b}}$ is replaced by $+\infty$ if $b \leq 0$. If $K_P^{\mathrm{rad}} \leq b$ in $B_P[R]$, then*

$$\sup_M \min \left\{ \max_{\sigma \subset W} K_f(\sigma) : \dim W > p + l \right\} \geq \max\{-b, 0\}.$$

Moreover,

$$\sup_M \min \left\{ \max_{\sigma \subset W} K_M(\sigma) : \dim W > p + l \right\} \geq \max\{-b, 0\} + \inf_{B_P[R]} K_P.$$

It is not clear the extent to which the above conjecture is true, but an affirmative answer at least in the most important case $P^n = \mathbb{R}^n$, $l = 0$, $p = m-1$ would provide the extension of Efimov's theorem to codimension $m-1$ and consequently settle the problem of isometric immersions $f : \mathbb{H}^m \to \mathbb{R}^{2m-1}$.

Remark 6.2. We said that Conjecture 6.1 was the limiting case of Theorem 4.1. However, we do not add hypothesis (4.2). Indeed, (4.2) is important only to ensure that the Omori-Yau maximum principle for the Hessian holds on M^m. This latter principle is one of our main tools to build the proof of Theorem 4.1, but the above conjecture seems to be inaccessible to techniques using it. Moreover, removing (4.2) allows us to include the aforementioned extension of Efimov's theorem as an important particular case of the conjecture.

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