Proceedings Book of International Workshop on Theory of Submanifolds (Volume: 1 (2016)) June 2–4, 2016, Istanbul, Turkey. Editors: Nurettin Cenk Turgay, Elif Özkara Canfes, Joeri Van der Veken and Cornelia-Livia Bejan
Recieved: January 17, 2017
Accepted: April 19, 2017
DOI: 10.24064/iwts2016.2017.5

Homothetic Motion and Surfaces with Pointwise 1-Type Gauss Map in $\mathbb{E}^4$

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Abstract. In this paper, we determine a surface M by means of homothetic motion in $\mathbb{E}^4$ and we give necessary and sufficient conditions for flat surface M with flat normal bundle to have pointwise 1-type Gauss map. Also, we show that flat surface M with flat normal bundle which have pointwise 1-type Gauss map of the first kind is a Clifford Torus. Morever, we obtain a characterization of minimal surface M with pointwise 1-type Gauss map.
Keywords. Homothetic motion · submanifolds · Gauss map · Pointwise 1-type Gauss map.
MSC 2010 Classification. Primary: 53B25; Secondary:53C40.

1 Introduction

A submanifold M of a Euclidean space $\mathbb{E}^m$ is said to be of finite type if its position vector x can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, that is, $x = x_0 + x_1 + ... + x_k$, where x_0 is a constant map, $x_1, ..., x_k$ are non-constant maps such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, ..., k$. If $\lambda_1, \lambda_2, ..., \lambda_k$ are all different, then M is said to be of $k-$type. This definition was similarly extended to differentiable maps, in particular, to Gauss maps of submanifolds [3].

If a submanifold M of a Euclidean space has 1-type Gauss map G, then G satisfies $\Delta G = \lambda (G + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector C. Chen and Piccinni made a general study on compact submanifolds of Euclidean spaces with finite type Gauss map and they proved that a compact hypersurface M of $\mathbb{E}^{n+1}$ has 1-type Gauss map if and only if M is a hypersphere in $\mathbb{E}^{n+1}$ [3].

Hovewer, the Laplacian of the Gauss map of some typical well known surfaces such as a helicoid, a catenoid and a right cone in Euclidean 3-space $\mathbb{E}^3$ take a some what different form, namely,

$$\Delta G = f (G + C) \tag{1.1}$$

for some smooth function f on M and some constant vector C. A submanifold M of a Euclidean space $\mathbb{E}^m$ is said to have pointwise 1-type Gauss map if its Gauss map satisfies (1) for some smooth function f on M and some constant vector C. A submanifold with pointwise 1-type Gauss map is said to be of the first kind if the vector C in (1) is zero vector. Otherwise, the pointwise 1-type Gauss map is said to be of the second kind. A pointwise 1-type Gauss map is called proper if the function f given by (1.1) is non-constant. Non-proper pointwise 1-type Gauss map is just usual 1-type Gauss map.

Surfaces in Euclidean space with pointwise 1-type Gauss map were recently studied in [4], [5], [6]. Also Dursun and Turgay in [7] gave all general rotational surfaces in $\mathbb{E}^4$ with proper pointwise 1-type Gauss map of the first kind and classified minimal rotational surfaces with proper pointwise 1-type Gauss map of the second kind. Arslan et al. in [1] investigated rotational embedded surface with pointwise 1-type Gauss map. Arslan at el. in [2] gave necessary and sufficent conditions for Vranceanu rotation surface to have pointwise 1-type Gauss map. Yoon in [8] showed that flat Vranceanu rotation surface with pointwise 1-type Gauss map is a Clifford torus.

In this paper, we determine a surface M by means of homothetic motion in $\mathbb{E}^4$ and we give necessary and sufficient conditions for flat surface M with flat normal bundle to have pointwise 1-type Gauss map. We show that flat surface with flat normal bundle which has pointwise 1-type Gauss map of the first kind is a Clifford Torus. Morever we obtain a characterization of minimal surface M with pointwise 1-type Gauss map.

2 Preliminaries

Let M be an oriented $n-$dimensional submanifold in $m-$dimensional Euclidean space $\mathbb{E}^m$. Let $e_1,...,e_n, e_{n+1},...,e_m$ be an oriented local orthonormal frame in $\mathbb{E}^m$ such that $e_1,...,e_n$ are tangent to M and $e_{n+1},...,e_m$ normal to M. We use the following convention on the ranges of indices: $1 \leq i, j, k,... \leq n$, $n+1 \leq r, s, t,... \leq m$, $1 \leq A, B, C,... \leq m$.

Let $\tilde{\nabla}$ be the Levi-Civita connection of $\mathbb{E}^m$ and ∇ the induced connection on M. Let ω_A be the dual-1 form of e_A defined by $\omega_A(e_B) = \delta_{AB}$. Also, the connection forms ω_{AB} are defined by

$$de_A = \sum_B \omega_{AB} e_B, \quad \omega_{AB} + \omega_{BA} = 0.$$

Then we have

$$\tilde{\nabla}_{e_k} e_i = \sum_{j=1}^{n} \omega_{ij}(e_k) e_j + \sum_{r=n+1}^{m} h_{ik}^r e_r$$

and

$$\tilde{\nabla}_{e_k} e_s = -A_s(e_k) + D_{e_k} e_s, \quad D_{e_k} e_s = \sum_{r=n+1}^{m} \omega_{sr}(e_k) e_r,$$

where D is the normal connection, h^r_{ik} the coefficients of the second fundamental form h and A_s the Weingarten map in the direction e_s.

For any real function f on M, the Laplacian of f is defined by

$$\Delta f = -\sum_i \left(\tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} f - \tilde{\nabla}_{\nabla^{e_i}_{e_i}} f \right). \tag{2.1}$$

The mean curvature vector H and Gaussian curvature K are defined by

$$H = \frac{1}{n} \sum_{r,i} h^r_{ii} e_r \tag{2.2}$$

and

$$K = \sum_{s=n+1}^{m} \left(h^s_{11} h^s_{22} - h^s_{12} h^s_{21} \right). \tag{2.3}$$

Also normal curvature tensor R^D of M in $\mathbb{E}^m$ is given by

$$R^D(e_j, e_k; e_r, e_s) = \sum_{i=1}^{n} \left(h^r_{ik} h^s_{ij} - h^r_{ij} h^s_{ik} \right). \tag{2.4}$$

Let us now define the Gauss map G of a submanifold M into $G(n,m)$ in $\wedge^n \mathbb{E}^m$, where $G(n,m)$ is the Grassmannian manifold consisting of all oriented $n-$planes through the origin of $\mathbb{E}^m$ and $\wedge^n \mathbb{E}^m$ is the vector space obtained by the exterior product of n vectors in $\mathbb{E}^m$. In a natural way, we can identify $\wedge^n \mathbb{E}^m$ with some Euclidean space $\mathbb{E}^N$ where $N = \begin{pmatrix} m \\ n \end{pmatrix}$. The map $G : M \to G(n,m) \subset E^N$ defined by $G(p) = (e_1 \wedge ... \wedge e_n)(p)$ is called the Gauss map of M, that is, a smooth map which carries a point p in M into the oriented $n-$plane through the origin of $\mathbb{E}^m$ obtained from parallel translation of the tangent space of M at p in $\mathbb{E}^m$.

The Laplacian of the Gauss map G for an $n-$dimensional submanifold M of Euclidean space $\mathbb{E}^m$ was given by

Lemma 2.1. (*See [3]*) *Let $x : M \to \mathbb{E}^m$ be an isometric immersion of an oriented n-dimensional Riemannian manifold M into $\mathbb{E}^m$. Then the Laplacian of the Gauss map $G : M \to G(n,m) \subset \wedge^n \mathbb{E}^m$ is given by*

$$\begin{aligned} \Delta G &= -n \sum_i e_1 \wedge ... \wedge D_{e_i} H \wedge ... \wedge e_n \\ &\quad + R^D(e_j, e_k; e_r, e_s)\, e_1 \wedge ... \wedge e_s^{k\ th} \wedge ... \wedge e_r^{j\ th} \wedge ... \wedge e_n + \|h\|^2 G. \end{aligned} \tag{2.5}$$

3 Homothetic Motion and Surfaces with Pointwise 1-Type Gauss Map

In this section, we define a surface by using the homothetic motion as follows:

$$f(t,s)=h(t)\begin{pmatrix}\cos t & -\sin t & 0 & 0\\ \sin t & \cos t & 0 & 0\\ 0 & 0 & \cos t & -\sin t\\ 0 & 0 & \sin t & \cos t\end{pmatrix}\begin{pmatrix}\alpha_1(s)\\ \alpha_2(s)\\ \alpha_3(s)\\ \alpha_4(s)\end{pmatrix}+\begin{pmatrix}C_1(t)\\ C_2(t)\\ C_3(t)\\ C_4(t)\end{pmatrix}, \tag{3.1}$$

where $h(t)$ is the homothetic scale of the motion, $C(t)=(C_1(t),C_2(t),C_3(t),C_4(t))$ is the translation vector and $\alpha(s)=(\alpha_1(s),\alpha_2(s),\alpha_3(s),\alpha_4(s))$ is a profile curve. If we choose the profile curve α as $\alpha(s)=(u(s)\cos s,0,u(s)\sin s,0)$ and the translation vector $C(t)=\vec{0}$ in (3.1), we obtain the surface M as follows:

$$f(s,t)=(u(s)h(t)\cos s\cos t,u(s)h(t)\cos s\sin t,u(s)h(t)\sin s\cos t,u(s)h(t)\sin s\sin t) \tag{3.2}$$

Let M be a surface in $\mathbb{E}^4$ given by the parametrization (3.2). The tangent vectors of $f(s,t)$ can be easily computed as

$$\begin{aligned}\vec{v}_1 &= \frac{\partial f}{\partial t}=(A_1B_1',A_1B_2',A_2B_1',A_2B_2'),\\ \vec{v}_2 &= \frac{\partial f}{\partial s}=\left(\dot{A}_1B_1,\dot{A}_1B_2,\dot{A}_2B_1,\dot{A}_2B_2\right)\end{aligned}$$

and a basis of the normal space of $f(s,t)$ can be given as follows:

$$\begin{aligned}\vec{v}_3 &= (-A_2B_2,A_2B_1,A_1B_2,-A_1B_1),\\ \vec{v}_4 &= \left(-\dot{A}_2B_2',\dot{A}_2B_1',\dot{A}_1B_2',-\dot{A}_1B_1'\right),\end{aligned}$$

where

$$\begin{aligned}A_1 &= u(s)\cos s,\quad A_2=u(s)\sin s\\ B_1 &= h(t)\cos t,\quad B_2=h(t)\sin t\end{aligned}$$

and $\dot{A}_i=\frac{\partial A_i}{\partial s}$ for $i=1,2$ and $B_j'=\frac{\partial B_j}{\partial t}$ $j=1,2$. By using Gramm-Schmidth orthonormalization, the orthonormal vectors of tangent and normal spaces of M are obtained, respectively, by

$$\begin{aligned}e_1 &= \frac{1}{\sqrt{v_{11}}}\vec{v}_1,\\ e_2 &= \frac{1}{\sqrt{|v_{11}(v_{11}v_{22}-v_{12}^2)|}}(v_{11}\vec{v}_2-v_{12}\vec{v}_1)\end{aligned}$$

and

$$
\begin{aligned}
e_3 &= \frac{1}{\sqrt{v_{33}}}\vec{v}_3, \\
e_4 &= \frac{1}{\sqrt{\left|v_{33}\left(v_{33}v_{44}-v_{34}^2\right)\right|}}\left(v_{33}\vec{v}_4 - v_{34}\vec{v}_3\right),
\end{aligned}
$$

where

$$
\begin{aligned}
v_{11} &= \langle \vec{v}_1, \vec{v}_1\rangle = u^2(s)\left(h^2(t) + (h'(t))^2\right), \\
v_{12} &= \langle \vec{v}_1, \vec{v}_2\rangle = u(s)\,\dot{u}(s)\,h(t)\,h'(t), \\
v_{22} &= \langle \vec{v}_2, \vec{v}_2\rangle = \left(u^2(s) + (\dot{u}(s))^2\right)h^2(t), \\
v_{33} &= \langle \vec{v}_3, \vec{v}_3\rangle = u^2(s)\,h^2(t), \\
v_{34} &= \langle \vec{v}_3, \vec{v}_4\rangle = u(s)\,\dot{u}(s)\,h(t)\,h'(t), \\
v_{44} &= \langle \vec{v}_4, \vec{v}_4\rangle = \left(u^2(s) + (\dot{u}(s))^2\right)\left(h^2(t) + (h'(t))^2\right).
\end{aligned}
$$

Hence, $\{e_1, e_2, e_3, e_4\}$ is orthonormal moving frame on M. Then we have the dual 1-forms as:

$$
\begin{aligned}
\omega_1 &= \frac{\dot{u}hh'}{\left(h^2 + (h')^2\right)^{\frac{1}{2}}}ds + \frac{u\left(h^2 + (h')^2\right)}{\left(h^2 + (h')^2\right)^{\frac{1}{2}}}dt \\
\omega_2 &= \frac{h\left(u^2h^2 + u^2(h')^2 + (\dot{u})^2h^2\right)^{\frac{1}{2}}}{\left(h^2 + (h')^2\right)^{\frac{1}{2}}}ds
\end{aligned}
$$

By a direct computation we have components of the second fundamental form and the connection forms as:

$$
h_{11}^3 = 0, \quad h_{12}^3 = -\frac{1}{W^{\frac{1}{2}}}, \quad h_{22}^3 = 2\frac{\dot{u}h'}{W} \tag{3.3}
$$

$$
\begin{aligned}
h_{11}^4 &= \frac{\left(2(h')^2 - hh'' + h^2\right)}{\left(h^2 + (h')^2\right)W^{\frac{1}{2}}}, \\
h_{12}^4 &= \frac{\dot{u}h'\left(hh'' - (h')^2\right)}{\left(h^2 + (h')^2\right)W}, \\
h_{22}^4 &= \frac{\left(2(\dot{u})^2 - u\ddot{u} + u^2\right)\left(h^2 + (h')^2\right)^2 - (\dot{u})^2(h')^2\left(hh'' + h^2\right)}{\left(h^2 + (h')^2\right)W^{\frac{3}{2}}}
\end{aligned} \tag{3.4}
$$

and

$$\begin{aligned}\omega_{12} &= -\frac{\dot{u}h\left(2\left(h'\right)^2 - hh'' + h^2\right)}{u\left(h^2 + \left(h'\right)^2\right)^{\frac{3}{2}} W^{\frac{1}{2}}}\omega_1 \\ &\quad + \frac{u^2 h'\left(h^2 + \left(h'\right)^2\right)^2 + \left(\dot{u}\right)^2 h^2 h'\left(2\left(h'\right)^2 - hh'' + h^2\right)}{uh\left(h^2 + \left(h'\right)^2\right)^{\frac{3}{2}} W}\omega_2, \qquad (3.5) \\ \omega_{34} &= \frac{\dot{u}h}{u\left(h^2 + \left(h'\right)^2\right)^{\frac{1}{2}} W^{\frac{1}{2}}}\omega_1 + \frac{h'\left(u^2h^2 + u^2\left(h'\right)^2 - \left(\dot{u}\right)^2 h^2\right)}{uh\left(h^2 + \left(h'\right)^2\right)^{\frac{1}{2}} W}\omega_2,\end{aligned}$$

where $W = u^2h^2 + u^2\left(h'\right)^2 + \left(\dot{u}\right)^2 h^2$.

Proposition 3.1. *Let M be the surface given by the parameterization (3.2). The Gaussian curvature and the normal bundle curvature of M are given, respectively, by*

$$K = \frac{\left(2\left(h'\right)^2 - hh'' + h^2\right)\left(2\left(\dot{u}\right)^2 - u\ddot{u} + u^2\right) - \left(h^2 + \left(h'\right)^2\right)\left(u^2 + \left(\dot{u}\right)^2\right)}{W^2} \qquad (3.6)$$

and

$$R^D = \frac{\left(2\left(\dot{u}\right)^2 - u\ddot{u} + u^2\right)\left(h^2 + \left(h'\right)^2\right) - \left(2\left(h'\right)^2 - hh'' + h^2\right)\left(u^2 + \left(\dot{u}\right)^2\right)}{W^2} \qquad (3.7)$$

Proof. By using (2.3), (2.4), (3.3) and (3.4), we obtain (3.6) and (3.7). □

Corollary 3.2. *Let M be the surface given by the parameterization (3.2). M is a flat surface with flat normal bundle if and only if it is parameterized by*

$$f(t,s) = a_1 a_2 e^{k_1 t + k_2 s}\left(\cos s\cos t, \cos s\sin t, \sin s\cos t, \sin s\sin t\right) \qquad (3.8)$$

or

$$f(t,s) = \frac{c_1 c_2}{\sqrt{\left|\cos\left(2t + b_1\right)\right|}\sqrt{\left|\cos\left(2s + b_2\right)\right|}}\left(\cos s\cos t, \cos s\sin t, \sin s\cos t, \sin s\sin t\right) \qquad (3.9)$$

Proof. Let M be a flat surface with flat normal bundle. Then both $K = 0$ and $R^D = 0$. From (3.6), we have

$$\frac{2\left(h'\right)^2 - hh'' + h^2}{h^2 + \left(h'\right)^2} . \frac{2\left(\dot{u}\right)^2 - u\ddot{u} + u^2}{u^2 + \left(\dot{u}\right)^2} = 1 \qquad (3.10)$$

and from (3.7), we get

$$\frac{2\left(h'\right)^2 - hh'' + h^2}{h^2 + \left(h'\right)^2} = \frac{2\left(\dot{u}\right)^2 - u\ddot{u} + u^2}{u^2\left(s\right) + \left(\dot{u}\right)^2}. \tag{3.11}$$

By combining (3.10) and (3.11) and solving these differential equations we obtain

$$h(t) = a_1 e^{k_1 t} \text{ and } u\left(s\right) = a_2 e^{k_2 s}$$

or

$$h(t) = \frac{c_1}{\sqrt{\left|\cos\left(2t + b_1\right)\right|}} \text{ and } u\left(s\right) = \frac{c_2}{\sqrt{\left|\cos\left(2s + b_2\right)\right|}},$$

where $a_1, a_2, b_1, b_2, c_1, c_2, k_1$ and k_2 are real constants. □

Remark 3.3. The surface M given by the parameterization (3.2) can be considered as the tensor product surface of two Euclidean planar curves, that is, let $\alpha : \mathbb{R} \to \mathbb{R}^2$, $\alpha(s) = (\alpha_1(s), \alpha_2(s))$ and $\beta : \mathbb{R} \to \mathbb{R}^2$, $\beta(t) = (\beta_1(t), \beta_2(t))$ be two Euclidean planar curves. The tensor product surface $f\left(t, s\right)$ is defined by

$$f = \alpha \otimes \beta : \mathbb{R}^2 \to \mathbb{R}^4,$$

$$f\left(t, s\right) = \left(\alpha_1(s)\beta_1(t), \alpha_1(s)\beta_2(t), \alpha_2(s)\beta_1(t), \alpha_2(s)\beta_2(t)\right).$$

In particular, for the curves $\alpha(s) = \left(u\left(s\right)\cos s, u\left(s\right)\sin s\right)$ and $\beta(t) = \left(h(t)\cos t, h(t)\sin t\right)$ the tensor product of them gives the surface M given by the parameterization (3.2).

Theorem 3.4. *(See [9]). A regular tensor product surface $x(s,t) = \alpha(s) \otimes \beta(t)$ of two curves $\alpha : \mathbb{R} \to \mathbb{R}^2$, $\alpha(s) = \left(u\left(s\right)\cos s, u\left(s\right)\sin s\right)$ or $\beta : \mathbb{R} \to \mathbb{R}^2$, $\beta(t) = \left(h(t)\cos t, h(t)\sin t\right)$ is flat if and only if either*

1. α or β is a straight line through the origin.

2. α and β are sinusoidal spirals, that is, the curves α and β are parameterized by

$$\begin{aligned} \alpha(s) &= c_1 \left|\cos\left(\left(a+1\right)s + b_1\right)\right|^{-\frac{1}{a+1}} \left(\cos s, \sin s\right) \\ \beta(t) &= c_2 \left|\cos\left(\left(\frac{1}{a} + 1\right)t + b_2\right)\right|^{-\frac{1}{\frac{1}{a}+1}} \left(\cos t, \sin t\right) \end{aligned}$$

3. α and β are logarithmic spirals, that is, the curves α and β are parameterized by

$$\alpha(s) = a_1 e^{k_1 s}\left(\cos s, \sin s\right) \text{ and } \beta(t) = a_2 e^{k_2 t}\left(\cos t, \sin t\right)$$

with $a_1, a_2, b_1, b_2, c_1, c_2, k_1$ and k_2 are real constants, $a_1, a_2, c_1, c_2 > 0$ and $a \neq -1$.

Remark 3.5. In [8] Yoon studied Vranceanu surface parameterized by

$$f(s,t) = (u(s)\cos s\cos t, u(s)\cos s\sin t, u(s)\sin s\cos t, u(s)\sin s\sin t).$$

He proved that flat Vranceanu surface in E^4 has pointwise 1-type Gauss map if and only if it is a Clifford torus. Also the normal bundle of flat Vranceanu surface is flat, too.

Now we investigate flat surface M with flat normal bundle with pointwise 1-type Gauss map.

Theorem 3.6. *Let M be flat surface with flat normal bundle given by the parameterization (3.2). Then M has pointwise 1-type Gauss map if and only if either*

1. M is a Clifford torus, that is, the product of two plane circles with same radius

2. It is the product of two logarithmic spirals which is parameterized by

$$f(t,s) = e^{k(t\pm s)}(\cos s\cos t, \cos s\sin t, \sin s\cos t, \sin s\sin t)$$

where k is non zero real constant.

Proof. Firstly, we assume that the flat surface M with flat normal bundle given by the parameterization (3.8) has pointwise 1-type Gauss map. If necessary, by an appropriate homothetic transformation we may assume that $a_1 = a_2 = 1$. Then we have $h(t) = e^{k_1 t}$ and $u(s) = e^{k_2 s}$. By using (3.3), (3.4) and (3.5) we have components of the second fundamental form and the connection forms as:

$$h_{11}^3 = 0, \quad h_{12}^3 = -\alpha(s,t), \quad h_{22}^3 = a\alpha(s,t)$$
$$h_{11}^4 = \alpha(s,t), \quad h_{12}^4 = 0, \quad h_{22}^4 = \alpha(s,t)$$

and

$$\begin{aligned} \omega_{12} &= b\alpha(s,t)\omega_1 + c\alpha(s,t)\omega_2, \quad \omega_{13} = -\alpha(s,t)\omega_2, \quad \omega_{14} = \alpha(s,t)\omega_1 \\ \omega_{23} &= -\alpha(s,t)\omega_1 + a\alpha(s,t)\omega_2 \quad \omega_{24} = \alpha(s,t)\omega_2, \quad \omega_{34} = -b\alpha(s,t)\omega_1 + d\alpha(s,t)\omega_2, \end{aligned}$$

By covariant differentiation with respect to e_1 and e_2 a straightforward calculation gives:

$$\begin{aligned} \tilde{\nabla}_{e_1} e_1 &= b\alpha e_2 + \alpha e_4, \\ \tilde{\nabla}_{e_2} e_1 &= c\alpha e_2 - \alpha e_3, \\ \tilde{\nabla}_{e_1} e_2 &= -b\alpha e_1 - \alpha e_3, \\ \tilde{\nabla}_{e_2} e_2 &= -c\alpha e_1 + a\alpha e_3 + \alpha e_4, \\ \tilde{\nabla}_{e_1} e_3 &= \alpha e_2 - b\alpha e_4, \\ \tilde{\nabla}_{e_2} e_3 &= \alpha e_1 - a\alpha e_2 + d\alpha e_4 \\ \tilde{\nabla}_{e_1} e_4 &= -\alpha e_1 + b\alpha e_3, \\ \tilde{\nabla}_{e_2} e_4 &= -\alpha e_2 - d\alpha e_3, \end{aligned} \tag{3.12}$$

where

$$
\begin{aligned}
\alpha(s,t) &= \frac{1}{u(s)h(t)\left(1+k_1^2+k_2^2\right)^{\frac{1}{2}}},\ a=\frac{2k_1k_2}{\left(1+k_1^2+k_2^2\right)^{\frac{1}{2}}},\ b=-\frac{k_2}{\left(1+k_1^2\right)^{\frac{1}{2}}},\\
c &= \frac{k_1\left(1+k_1^2+k_2^2\right)^{\frac{1}{2}}}{\left(1+k_1^2\right)^{\frac{1}{2}}},\ d=\frac{k_1\left(1+k_1^2-k_2^2\right)}{\left(1+k_1^2\right)^{\frac{1}{2}}\left(1+k_1^2+k_2^2\right)^{\frac{1}{2}}} \qquad (3.13)
\end{aligned}
$$

By using (2.1) and (3.12) and after straight-forward computations, the Laplacian ΔG of the Gauss map G can be expressed as

$$
\begin{aligned}
\Delta G &= \left(4+a^2\right)\alpha^2 e_1\wedge e_2+(c+d)\,\alpha^2 e_1\wedge e_3-(2b+ad)\,\alpha^2 e_1\wedge e_4\\
&\quad+(2b-ac)\,\alpha^2 e_2\wedge e_3-(c+d)\,\alpha^2 e_2\wedge e_4. \qquad (3.14)
\end{aligned}
$$

We suppose that the flat surface M with flat normal bundle has pointwise 1-type Gauss map. From (1.1) and (3.14), we get

$$\left(4+a^2\right)\alpha^2=f+f\left\langle C,e_1\wedge e_2\right\rangle \qquad (3.15)$$

$$(c+d)\,\alpha^2=f\left\langle C,e_1\wedge e_3\right\rangle \qquad (3.16)$$

$$(-2b-ad)\,\alpha^2=f\left\langle C,e_1\wedge e_4\right\rangle \qquad (3.17)$$

$$(2b-ac)\,\alpha^2=f\left\langle C,e_2\wedge e_3\right\rangle \qquad (3.18)$$

$$-(c+d)\,\alpha^2=f\left\langle C,e_2\wedge e_4\right\rangle \qquad (3.19)$$

Then, we have

$$\left\langle C,e_3\wedge e_4\right\rangle=0 \qquad (3.20)$$

By differentiating (3.20) with respect to e_1, we get

$$\left\langle C,e_1\wedge e_3\right\rangle+\left\langle C,e_2\wedge e_4\right\rangle=0 \qquad (3.21)$$

When we take the derivative of (3.20) with respect to e_2, we have

$$\left\langle C,e_1\wedge e_4\right\rangle+\left\langle C,e_2\wedge e_3\right\rangle-a\left\langle C,e_2\wedge e_4\right\rangle=0 \qquad (3.22)$$

If we evaluate the derivative of (3.22) with respect to e_2 again, we get

$$
\begin{aligned}
2\left\langle C,e_1\wedge e_2\right\rangle &= -(c+d)\left\langle C,e_1\wedge e_3\right\rangle+ac\left\langle C,e_1\wedge e_4\right\rangle \qquad (3.23)\\
&\quad+ad\left\langle C,e_2\wedge e_3\right\rangle+(c+d)\left\langle C,e_2\wedge e_4\right\rangle
\end{aligned}
$$

By using (3.15), (3.16), (3.17), (3.18), (3.19), (3.21) and (3.23) we then have

$$f=\left(4+a^2+(c+d)^2+abc+a^2cd-abd\right)\alpha^2=A\alpha^2 \qquad (3.24)$$

that is, a smooth function f depends on s and t. Differentiating (3.24) with respect to e_1, we have

$$e_1(f)=-2cA\alpha^3. \qquad (3.25)$$

On the other hand, by differentiating (3.19) with respect to e_1 and by using (3.12), (3.15), (3.17), (3.18), (3.19), (3.24) and (3.25) we obtain

$$4b^2 + 2abd - 2abc - a^2cd - (c+d)^2 = 0. \tag{3.26}$$

By substituting (3.13) into (3.26) we get

$$\left(k_1^2 - k_2^2\right)\left(1 + k_1^2 + k_2^2 + k_1^2k_2^2\right) = 0 \tag{3.27}$$

and from (3.27) we obtain that $k_1 = \pm k_2$. In particular, if we take as $k_1 = k_2 = 0$, we obtain Clifford torus. For the other cases, we obtain the tensor product surface of two logarithmic spirals.

Conversely, we assume that $k_1^2 = k_2^2$. In that case the flat surface M with flat normal bundle is given by the parametrization (3.8) has pointwise 1-type Gauss map for the function

$$f(s,t) = \Big(4 + a^2 + (c+d)^2 + abc + a^2cd - abd\Big)\alpha^2 = A\alpha^2$$

and the constant vector

$$\begin{aligned} C =& \frac{1}{A}\big(\left(4 + a^2 - A\right)e_1 \wedge e_2 + (c+d)\,e_1 \wedge e_3 - (2b + ad)\,e_1 \wedge e_4\big) \\ &+ \frac{1}{A}\big((2b - ac)\,e_2 \wedge e_3 - (c+d)\,e_2 \wedge e_4\big). \end{aligned}$$

Now, we assume that the flat surface M with flat normal bundle is given by the parametrization (3.9). We research whether this surface has pointwise 1-type Gauss map. We can write as

$$u(s) = c_1\left(\varepsilon\cos(2s)\right)^{-\frac{1}{2}},$$

where if $\cos(2s) > 0$ (resp. < 0), then $\varepsilon = 1$ (resp. $= -1$). Analogously, we can write as

$$h(t) = c_2\left(\delta\cos(2t)\right)^{-\frac{1}{2}},$$

where if $\cos(2t) > 0$ (< 0, respectively) then $\delta = 1(-1$,respectively). By using (3.3), (3.4) and (3.5) we have components of the second fundamental form and the connection forms as:

$$h_{11}^3 = 0, \quad h_{12}^3 = -\lambda(s,t), \quad h_{22}^3 = \varkappa(s,t)\lambda(s,t)$$

$$h_{11}^4 = -\lambda(s,t), \quad h_{12}^4 = \varkappa(s,t)\lambda(s,t), \quad h_{22}^4 = -\left(1 + \varkappa^2(s,t)\right)\lambda(s,t)$$

and

$$\begin{aligned} \omega_{12} &= \tau(s,t)\,\lambda(s,t)\,\omega_1 + \beta(s,t)\,\lambda^2(s,t)\,\omega_2 \\ \omega_{34} &= \tau(s,t)\,\lambda(s,t)\,\omega_1 + \beta(s,t)\,\lambda^2(s,t)\,\omega_2. \end{aligned}$$

By covariant differentiation with respect to e_1 and e_2, we get

$$\begin{aligned}
\tilde{\nabla}_{e_1} e_1 &= \tau\lambda e_2 - \lambda e_4, \\
\tilde{\nabla}_{e_2} e_1 &= \beta\lambda^2 e_2 - \lambda e_3 + \varkappa\lambda e_4, \\
\tilde{\nabla}_{e_1} e_2 &= -\tau\lambda e_1 - \lambda e_3 + \varkappa\lambda e_4, \\
\tilde{\nabla}_{e_2} e_2 &= -\beta\lambda^2 e_1 + \varkappa\lambda e_3 - \left(1+\varkappa^2\right)\lambda e_4, \\
\tilde{\nabla}_{e_1} e_3 &= \lambda e_2 + \tau\lambda e_4, \\
\tilde{\nabla}_{e_2} e_3 &= \lambda e_1 - \varkappa\lambda e_2 + \beta\lambda^2 e_4 \\
\tilde{\nabla}_{e_1} e_4 &= \lambda e_1 - \varkappa\lambda e_2 - \tau\lambda e_3, \\
\tilde{\nabla}_{e_2} e_4 &= -\varkappa\lambda e_1 + \left(1+\varkappa^2\right)\lambda e_2 - \beta\lambda^2 e_3,
\end{aligned} \tag{3.28}$$

where

$$\begin{aligned}
\varkappa(s,t) &= \frac{2\left(\varepsilon\sin(2s)\right)\left(\delta\sin(2t)\right)}{\left(1-\left(\varepsilon\sin(2s)\right)^2\left(\delta\sin(2t)\right)^2\right)^{\frac{1}{2}}}, \\
\tau(s,t) &= \frac{\left(\varepsilon\sin(2s)\right)\left(\delta\cos(2t)\right)}{\left(\varepsilon\cos(2s)\right)}, \\
\beta(s,t) &= \frac{c_1c_2\left(\delta\sin(2t)\right)\left(\left(\varepsilon\cos(2s)\right)^2-\left(\varepsilon\sin(2s)\right)^2\left(\delta\cos(2t)\right)^2\right)}{\left(\varepsilon\cos(2s)\right)^{\frac{5}{2}}\left(\delta\cos(2t)\right)^{\frac{5}{2}}}, \\
\lambda(s,t) &= \frac{1}{W^{\frac{1}{2}}} = \frac{\left(\varepsilon\cos(2s)\right)^{\frac{3}{2}}\left(\delta\cos(2t)\right)^{\frac{3}{2}}}{c_1c_2\left(1-\left(\varepsilon\sin(2s)\right)^2\left(\delta\sin(2t)\right)^2\right)^{\frac{1}{2}}}.
\end{aligned}$$

By using (2.1), straight-forward computation the Laplacian ΔG of the Gauss map G can be expressed as

$$\begin{aligned}
\Delta G = {} & \left(4+5\varkappa^2+\varkappa^4\right)\lambda^2 e_1\wedge e_2 + \left(-e_2\left(\varkappa\lambda\right) - \beta\left(2+\varkappa^2\right)\lambda^3\right)e_1\wedge e_3 \\
& + \left(e_2\left(\left(2+\varkappa^2\right)\lambda\right) - \beta\varkappa\lambda^3\right)e_1\wedge e_4 \\
& + \left(e_1\left(\varkappa\lambda\right) + \tau\left(2+\varkappa^2\right)\lambda^2\right)e_2\wedge e_3 + \left(-e_1\left(\left(2+\varkappa^2\right)\lambda\right) + \tau\varkappa\lambda^2\right)e_2\wedge e_4.
\end{aligned} \tag{3.29}$$

We suppose that the flat surface M with flat normal bundle has pointwise 1-type Gauss map. From (1.1) and (3.29), we get

$$\left(4+5\varkappa^2+\varkappa^4\right)\lambda^2 = f + f\left\langle C, e_1\wedge e_2\right\rangle, \tag{3.30}$$

$$-e_2\left(\varkappa\lambda\right) - \beta\left(2+\varkappa^2\right)\lambda^3 = f\left\langle C, e_1\wedge e_3\right\rangle, \tag{3.31}$$

$$e_2\left(\left(2+\varkappa^2\right)\lambda\right) - \beta\varkappa\lambda^3 = f\left\langle C, e_1\wedge e_4\right\rangle, \tag{3.32}$$

$$e_1\left(\varkappa\lambda\right) + \tau\left(2+\varkappa^2\right)\lambda^2 = f\left\langle C, e_2\wedge e_3\right\rangle, \tag{3.33}$$

$$-e_1\left(\left(2+\varkappa^2\right)\lambda\right) + \tau\varkappa\lambda^2 = f\left\langle C, e_2\wedge e_4\right\rangle. \tag{3.34}$$

Then we have

$$\left\langle C, e_3\wedge e_4\right\rangle = 0. \tag{3.35}$$

By differentiating (3.35) with respect to e_1, we get

$$\langle C, e_2 \wedge e_4\rangle - \langle C, e_1 \wedge e_3\rangle + \varkappa \langle C, e_2 \wedge e_3\rangle = 0. \tag{3.36}$$

By considering together with (3.31), (3.33), (3.34) and (3.36), we have

$$\begin{aligned} -e_1\left(\left(2+\varkappa^2\right)\lambda\right) + \tau\varkappa\lambda^2 + \varkappa\left(e_1(\varkappa\lambda) + \tau\left(2+\varkappa^2\right)\lambda^2\right) \\ + e_2(\varkappa\lambda) + \beta\left(2+\varkappa^2\right)\lambda^3 = 0. \end{aligned} \tag{3.37}$$

On the other hand, after some long computations we have

$$e_1(\varkappa) = \frac{4(\varepsilon\sin(2s))(\varepsilon\cos(2s))^{\frac{1}{2}}(\delta\cos(2t))^{\frac{5}{2}}}{c_1c_2\left(1-(\varepsilon\sin(2s))^2(\delta\sin(2t))^2\right)^{\frac{3}{2}}}, \tag{3.38}$$

$$\begin{aligned} e_2(\varkappa) =& \frac{4\lambda(\varepsilon\cos(2s))(\delta\sin(2t))(\delta\cos(2t))^{-1}}{\left(1-(\varepsilon\sin(2s))^2(\delta\sin(2t))^2\right)^{\frac{3}{2}}} \\ &- \frac{4\lambda(\varepsilon\sin(2s))^2(\varepsilon\cos(2s))^{-1}(\delta\sin(2t))(\delta\cos(2t))}{\left(1-(\varepsilon\sin(2s))^2(\delta\sin(2t))^2\right)^{\frac{3}{2}}}, \end{aligned} \tag{3.39}$$

$$\begin{aligned} e_1(\lambda) = \frac{((\varepsilon\cos(2s))^2(\delta\sin(2t))(\delta\cos(2t))^2)}{c_1^2c_2^2(1-(\varepsilon\sin(2s))^2(\delta\sin(2t))^2)^{\frac{3}{2}}}\Big(-3+2(\varepsilon\sin(2s))^2 \\ + (\varepsilon\sin(2s))^2(\delta\sin(2t))^2\Big) \end{aligned} \tag{3.40}$$

and

$$\begin{aligned} e_2(\lambda) =& \zeta\Big((-3+2(\delta\sin(2t))^2 + (\varepsilon\sin(2s))^2(\delta\sin(2t))^2) \\ &- (\delta\sin(2t))^2(-3+2(\varepsilon\sin(2s))^2 + (\varepsilon\sin(2s))^2(\delta\sin(2t))^2)\Big), \\ \zeta =& \frac{\lambda(\varepsilon\sin(2s)(\varepsilon\cos(2s))^{\frac{1}{2}}(\delta\cos(2t))^{\frac{1}{2}})}{c_1c_2(1-(\varepsilon\sin(2s))^2(\delta\sin(2t))^2)^{\frac{3}{2}}}. \end{aligned} \tag{3.41}$$

By combining (3.38), (3.39), (3.40) and (3.41) with (3.37), we obtain that this equation is not satisfied. So, there is no flat surface with flat normal bundle given by the parameterization (3.9) which has pointwise 1-type Gauss map. □

Corollary 3.7. *Let M be flat surface with flat normal bundle given by the parameterization (3.2). M has pointwise 1-type Gauss map of the first kind if and only if it is a Clifford Torus.*

Proof. From Theorem 3.6 the flat surface M with flat normal bundle is given by the parameterization (3.2) has pointwise 1-type Gauss map for the function

$$f(s,t) = A\alpha^2$$

and the constant vector

$$C = \frac{1}{A}\Big((4+a^2-A)e_1\wedge e_2 + (c+d)e_1\wedge e_3 - (2b+ad)e_1\wedge e_4\Big) + \frac{1}{A}\Big((2b-ac)e_2\wedge e_3 - (c+d)e_2\wedge e_4\Big)$$

with $k_1^2 = k_2^2$, where

$$A = \Big(4 + a^2 + (c+d)^2 + abc + a^2cd - abd\Big).$$

We assume that the surface M has pointwise 1-type Gauss map of the first kind. Then, we obtain $C = 0$, that is, all components of C is zero. Then, we get $k_1 = k_2 = 0$. This completes the proof. □

Theorem 3.8. *An oriented minimal surface M in the Euclidean space $\mathbb{E}^4$ has pointwise 1-type Gauss map of the first kind if and only if M has a flat normal bundle [6].*

Theorem 3.9. *There exists no minimal surface given by the parameterization (3.2) with pointwise 1-type Gauss map of the first kind.*

Proof. We suppose that the surface M given by the parameterization (3.2) is minimal surface with pointwise 1-type Gauss map of the first kind. From Theorem 3.8 we have $R^D = 0$. Since the surface M is minimal and its normal bundle is flat then (2.2) and (2.4) imply, respectively

$$h_{11}^3 + h_{22}^3 = 0 \text{ and } h_{11}^4 + h_{22}^4 = 0 \tag{3.42}$$

$$h_{12}^3\left(h_{11}^4 - h_{22}^4\right) + h_{12}^4\left(h_{22}^3 - h_{11}^3\right) = 0. \tag{3.43}$$

By combining (3.3), (3.4), (3.42) and (3.43) we have

$$h_{22}^3 = h_{11}^4 = h_{22}^4 = 0. \tag{3.44}$$

The equation (3.44) conflicts with the regularity of the surface. □

Theorem 3.10. *(See [6]). A non-planar minimal oriented surface M in the Euclidean space E^4 has pointwise 1-type Gauss map of the second kind if and only if, with respect to some suitable local orthonormal frame $\{e_1, e_2, e_3, e_4\}$ on M, the shape operators of M are given by*

$$A_3 = \begin{pmatrix} \rho & 0 \\ 0 & -\rho \end{pmatrix} \text{ and } A_4 = \begin{pmatrix} 0 & \varepsilon\rho \\ \varepsilon\rho & 0 \end{pmatrix},$$

where $\varepsilon = \pm 1$ and ρ is a smooth non-zero function on M.

Theorem 3.11. *Let M be minimal surface given by the parameterization (3.2). Then M has pointwise 1-type Gauss map of the second kind if and only if it is parametrized by*

$$f(t,s) = \frac{bd}{\sqrt{|\cos(2s+c)|}}(\cos s\cos t, \cos s\sin t, \sin s\cos t, \sin s\sin t)$$

or

$$f(t,s) = \frac{bd}{\sqrt{|\cos(2t+c)|}}(\cos s\cos t, \cos s\sin t, \sin s\cos t, \sin s\sin t)$$

where b, d and c are real constants.

Proof. We assume that M is a minimal surface with pointwise 1-type Gauss map of second kind. In that case the mean curvature of M is zero and we have

$$h_{22}^3 = 0 \tag{3.45}$$

and

$$h_{11}^4 + h_{22}^4 = 0. \tag{3.46}$$

By using (2.5), the Laplacian ΔG of the Gauss map G is written as

$$\Delta G = \|h\|^2 G + 2R^D e_3 \wedge e_4, \tag{3.47}$$

where $R^D \neq 0$. In the opposite case, M has pointwise 1-type Gauss map of the first kind. By using (2.4), (3.45) and (3.46) we get

$$R^D = 2h_{12}^3 h_{11}^4 \neq 0. \tag{3.48}$$

Since M has pointwise 1-type Gauss map of the second kind, from (1.1) and (3.47) we have

$$\|h\|^2 G + 2R^D e_3 \wedge e_4 = fG + fC \tag{3.49}$$

for some smooth non-zero function f on M and some constant vector C. Since the vector C is a linear combination of $e_1 \wedge e_2$, $e_1 \wedge e_3$, $e_1 \wedge e_4$, $e_2 \wedge e_3$, $e_2 \wedge e_4$, $e_3 \wedge e_4$. From (3.49) we get

$$\|h\|^2 = f\left(1 + \langle C, e_1 \wedge e_2\rangle\right) \tag{3.50}$$

$$2R^D = f\langle C, e_3 \wedge e_4\rangle \neq 0 \tag{3.51}$$

and

$$\langle C, e_1 \wedge e_3\rangle = \langle C, e_1 \wedge e_4\rangle = \langle C, e_2 \wedge e_3\rangle = \langle C, e_2 \wedge e_4\rangle = 0$$

Since h_{12}^3 is not equal to zero on M, it follows that$\|h\| \neq 0$ or $\langle C, e_1 \wedge e_2\rangle \neq -1$. Differentiating $\langle C, e_1 \wedge e_3\rangle = 0$ with respect to e_1 and e_2, we get

$$h_{12}^3\langle C, e_1 \wedge e_2\rangle + h_{11}^4\langle C, e_3 \wedge e_4\rangle = 0 \tag{3.52}$$

and

$$h_{12}^4\langle C, e_3 \wedge e_4\rangle = 0, \tag{3.53}$$

respectively. On the other hand, differentiating $\langle C, e_1 \wedge e_4\rangle = 0$ with respect to e_1 and e_2, we have

$$h_{12}^4\langle C, e_1 \wedge e_2\rangle = 0 \tag{3.54}$$

and

$$h_{12}^3\langle C, e_3 \wedge e_4\rangle + h_{11}^4\langle C, e_1 \wedge e_2\rangle = 0, \tag{3.55}$$

respectively. The equation (3.54) implies that $h_{12}^4 = 0$ or $\langle C, e_1 \wedge e_2 \rangle = 0$. If $\langle C, e_1 \wedge e_2 \rangle = 0$ then from (3.55) we get $h_{12}^3 \langle C, e_3 \wedge e_4 \rangle = 0$. h_{12}^3 is not equal to zero on M. Hence we have $\langle C, e_3 \wedge e_4 \rangle = 0$ and (3.51) implies that $R^D = 0$. This is a contradiction. So $\langle C, e_1 \wedge e_2 \rangle \neq 0$ and $h_{12}^4 = 0$. By using (3.52) and (3.55) we obtain

$$\left(h_{12}^3\right)^2 = \left(h_{11}^4\right)^2. \tag{3.56}$$

From (3.45) and (3.3) we get $\dot{u} = 0$ or $h' = 0$. Firstly we assume that $h' \neq 0$. Then we have $u = d$ =constant. By considering together with (3.3), (3.4), (3.46) and (3.56) we obtain

$$h(t) = \frac{c}{\sqrt{|\cos(2t+b)|}}.$$

Now we assume that $\dot{u} \neq 0$. Then we have $h = d =$ constant. By using (3.3) and (3.4) with $h = d$, we can see that (3.56) is satisfied directly. So, if we consider (3.4), (3.46) for $h = d$ we obtain

$$u(s) = \frac{c}{\sqrt{|\cos(2s+b)|}},$$

where b, c and d are real constants.

If we consider as both $\dot{u} = 0$ and $h' = 0$, then the surface M is not minimal surface.

On the other hand by using (3.47), (3.48), (3.50), (3.51), (3.55) and (3.56) (or see the proof of Theorem 5 in [6]) we can find the function f and the constant vector C as

$$f(s) = 8\left(h_{12}^3\right)^2 \tag{3.57}$$

and

$$C = -\frac{e_1 \wedge e_2}{2} + \varepsilon \frac{e_3 \wedge e_4}{2}. \tag{3.58}$$

Hence the minimal surface M has pointwise 1-type Gauss map of the second kind for the function f and the constant vector C given by (3.57) and (3.58), respectively. This completes the proof. □

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