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Hyper-Generalized Quasi Einstein Manifolds Satisfying Certain Ricci Conditions

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Abstract. In the present paper, we deal with hyper-generalized quasi Einstein manifold. First, we investigate geometric properties of this manifold with respect to its generators. Then, we study some classes of hyper-generalized quasi Einstein manifold satisfying certain Ricci conditions. Precisely, we obtain some necessary conditions for hyper-generalized quasi Einstein manifold to be a generalized quasi Einstein or a quasi Einstein manifold.
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1 Introduction

The notion of hyper-generalized quasi Einstein manifold has been first introduced by A. A. Shaikh, C. Özgür and A. Patra, in 2011 [18]. An n-dimensional Riemannian manifold (M^n, g), $(n > 2)$ is called a hyper-generalized quasi Einstein manifold if its Ricci tensor of type $(0,2)$ is non-zero and satisfies the following condition [2]

$$\begin{aligned} S(X,Y) =& ag(X,Y) + bA(X)A(Y) + c[A(X)B(Y) + A(Y)B(X)] \\ &+ d[A(X)D(Y) + A(Y)D(X)] \end{aligned} \tag{1.1}$$

for all $X, Y \in \chi(M)$, where a, b, c and d are real valued, non-zero scalar functions on (M^n, g), A, B and D are non-zero 1-forms such that

$$g(X,\rho_1) = A(X), \quad g(X,\rho_2) = B(X), \quad g(X,\rho_3) = D(X), \tag{1.2}$$

where ρ_1, ρ_2 and ρ_3 are three unit vector fields mutually orthogonal to each other at every point on M. The scalars a, b, c and d are called associated scalars of the manifold, A, B and D are called associated 1-forms and ρ_1, ρ_2 and ρ_3 are called the generators of the manifold. Throughout this work, such an n-dimensional manifold will be denoted by $(HGQE)_n$.

The name "hyper" is used as in the case of hyper-real numbers. Especially, if ρ_2 and ρ_3 are linearly dependent or if $d = 0$, then the notion of hyper-generalized quasi Einstein manifold turns into the notion of generalized quasi Einstein manifold introduced by M.C. Chaki in 2001, [2]. In [2, 6, 10], many authors studied this kind of manifolds. Recently, in [11], the authors obtained some properties of the generalized quasi Einstein manifolds satisfying some curvature conditions on the conformal, concircular, projective and the quasi-conformal curvature tensors. Furthermore, in [12], the authors investigated some geometric and physical properties of the generalized quasi Einstein manifolds with applications in general relativity. In addition to these studies, in [17] a non-trivial example of $(HGQE)_{2n+1}$ was given by Shaikh and Matsuyama which can be briefly summarized as follows:

Example 1.1. Let M^{2n+1} be an almost contact metric manifold [1] admitting an $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying

$$\phi\xi = 0, \qquad \eta \circ \phi = 0, \qquad \phi^2 = -I + \eta \otimes \xi, \tag{1.3}$$

$$g(\phi X, Y) = -g(X, \phi Y), \qquad \eta(X) = g(X, \xi), \qquad \eta(\xi) = 1, \tag{1.4}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{1.5}$$

for all vector fields $X, Y \in M^{2n+1}$ and such kind of manifold is denoted by $M^{2n+1}(\phi, \xi, \eta, g)$.

An almost contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is said to be a trans-Sasakian manifold [13] if the following condition holds:

$$(\nabla_X \phi)(Y) = \alpha[g(X, Y)\xi - \eta(Y)X] + \beta[g(\phi X, Y)\xi - \eta(Y)\phi X], \tag{1.6}$$

where α, β are smooth functions on M and such a structure is called *trans-Sasakian structure of type* (α, β). In [17], it was shown that in a conformally flat trans-Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g)$ of type (α, β), the Ricci tensor is of the form:

$$\begin{aligned} S(X, Y) =& ag(X, Y) + b\eta(X)\eta(Y) + c[\eta(X)\omega(Y) + \eta(Y)\omega(X)] \\ &+ d[\eta(X)\pi(Y) + \eta(Y)\pi(X)], \end{aligned} \tag{1.7}$$

where a, b, c and d are non-zero scalars given by

$$a = \frac{r}{2n} - (\alpha^2 - \beta^2), \quad b = -\frac{r}{2n} + (2n+1)(\alpha^2 - \beta^2), \quad c = 1, \quad d = -(2n-1) \tag{1.8}$$

and η, ω and π are non-zero 1-forms such that $\eta(X) = g(X, \xi)$ so,

$$\omega(X) = -((\phi X)\alpha) = g(X, \phi(grad\alpha)), \quad \pi(X) = (X\beta) = g(X, grad\beta) \tag{1.9}$$

for all X. Here, ξ is always orthogonal to $grad\beta$ and $\phi(grad\alpha)$ so we may only consider $grad\beta$ and $\phi(grad\alpha)$ are orthogonal to each other. Then, such a trans-Sasakian manifold is a $(HGQE)_{2n+1}$, which is neither $(QE)_{2n+1}$ nor $G(QE)_{2n+1}$, [17].

Let $\{e_i : i = 1, 2, \cdots, n\}$ be an orthonormal frame field at any point of the manifold. Then, setting $X = Y = e_i$ in (1.1) and taking summation over i; $(1 \leq i \leq n)$, we obtain

$$r = an + b, \tag{1.10}$$

where r is the curvature of the manifold. In view of the equations (1.1) and (1.2), in a hyper-generalized quasi Einstein manifold, we have

$$S(X, \rho_1) = (a + b)A(X) + cB(X) + dD(X), \tag{1.11}$$

$$S(X, \rho_2) = aB(X) + cA(X), \quad S(X, \rho_3) = aD(X) + dA(X), \tag{1.12}$$

$$S(\rho_1, \rho_1) = (a + b), \quad S(\rho_2, \rho_2) = S(\rho_3, \rho_3) = a, \quad S(\rho_1, \rho_2) = c, \quad S(\rho_1, \rho_3) = d. \tag{1.13}$$

If $d = c = 0$ in the fundamental equation (1.1) of $(HGQE)_n$, then the manifold reduces to a quasi Einstein manifold. Quasi Einstein manifolds have been studied by several authors, such as M.C. Chaki [3], U.C. De and G. C. Ghosh [5] and S. Guha [10]. Also, in [14]; A.A. Shaikh, Y.H. Kim and S.K. Hui and in [4]; A. De, C. Özgür and U.C. De studied on Lorentzian quasi Einstein spacetimes.

Similarly, if $d = c = b = 0$ in (1.1), then the manifold reduces to an Einstein manifold which is characterized by the proportionality of the Ricci tensor to the metric tensor.

Let R denote the Riemannian curvature tensor of M. The k-nullity distribution $N(k)$ [19] of a Riemannian manifold M is defined by the set of all vector fields $Z \in T_p(M)$ satisfying the condition $R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]$, for all $X, Y \in T_p(M)$ where k is some smooth function on M. In a quasi Einstein manifold M, if the generator U belongs to some k-nullity distribution, then M said to be an $N(k)$-quasi Einstein manifold [20]. C. Özgür and M. M. Tripathi [15] proved that in an n-dimensional $N(k)$-quasi Einstein manifold, $k = \frac{a+b}{n-1}$.

The importance of these manifolds is in fact due to the existence of certain spacetimes endowed with semi-Riemannian metrics. In general relativity and cosmology, the purpose of studying various types of semi-Riemannian manifolds is to represent the different phases in the evolution of the universe. Quasi Einstein spacetimes arose during the study of exact solutions of Einstein's field equations. For instance, the Robertson-Walker spacetimes are quasi Einstein spacetimes. While $(QE)_4$ can be taken as a model of perfect fluid spacetime, the importance of $G(QE)_4$ lies in the fact that such a 4-dimensional semi-Riemannian manifold is related to the study of general relativistic fluid spacetime admitting heat flux [16, 5]. Thus, the investigations on these manifolds with Riemannian or semi-Riemannian metric are very important in differential geometry as well as in general relativity and cosmology.

In this direction, this paper is organized as follows: First, in Section 2 we investigate geometric properties of $(HGQE)_n$ with respect to its generators. Then, some pseudo-symmetry types of such manifolds are considered and some necessary conditions for hyper-generalized quasi Einstein manifold to be a generalized quasi Einstein or a quasi Einstein manifold are obtained.

2 Some Geometric Properties of $(HGQE)_n$

In this section, we investigate some geometric properties of $(HGQE)_n$. First, we consider a $(HGQE)_n$ whose generator ρ_1 is a parallel vector field, that is

$$\nabla_X \rho_1 = 0, \tag{2.1}$$

where ∇ is the Levi-Civita connection. Then for all X and Y clearly we have, $(\nabla_X A)(Y) = g(\nabla_X \rho_1, Y) = 0$ and so $R(X,Y)\rho_1 = 0$. Contracting the last equation we get, $S(X, \rho_1) = 0$. Combining the last equation with (1.11), we obtain

$$(a+b)A(X) + cB(X) + dD(X) = 0. \tag{2.2}$$

Putting $X = \rho_1$ in (2.2), we get $a + b = 0$, putting $X = \rho_2$ in (2.2), we get $c = 0$ and putting $X = \rho_3$ in (2.2), we get $d = 0$ which means that such a manifold reduces to a quasi Einstein manifold and the sum of its associated scalar functions is zero.

On the other hand, if we assume that the generator ρ_2 is parallel vector field, then similarly we have $\nabla_X \rho_2 = 0$ and so $S(X, \rho_2) = 0$. Thus, in view of (1.11), we get

$$aB(X) + cA(X) = 0. \tag{2.3}$$

Putting $X = \rho_2$ in (2.3), we have $a = 0$ which is a contradiction. In a similar manner, if we assume that the generator ρ_3 is parallel vector field, then we have $\nabla_X \rho_3 = 0$ and so $S(X, \rho_3) = 0$ so, in view of (1.12), we get

$$aD(X) + dA(X) = 0. \tag{2.4}$$

Putting $X = \rho_3$ in (2.3), again we have $a = 0$. Hence we can state that:

Theorem 2.1. *In a $(HGQE)_n$, if the generator ρ_1 is a parallel vector field, then this manifold reduces to a quasi Einstein manifold in which the sum of its associated scalar functions is zero. But, the generators ρ_2 and ρ_3 can not be parallel vector fields.*

From Theorem (2.1), the Ricci tensor of the $(HGQE)_n$, whose generator ρ_1 is a parallel vector field, can be expressed as follows:

$$S(X,Y) = a[g(X,Y) - A(X)A(Y)]. \tag{2.5}$$

Taking the covariant derivative of the Ricci tensor and using the fact that ρ_1 is a parallel vector field, we obtain

$$(\nabla_Z S)(X,Y) = Z(a)[g(X,Y) - A(X)A(Y)]. \tag{2.6}$$

Contracting (2.6) over X and Z and using contracted second Bianchi Identity, we get

$$\frac{1}{2}Y(r) = Y(a) - \rho_1(a)A(Y). \tag{2.7}$$

Since $r = (n-1)a$, from (2.7) we obtain

$$(\frac{n-3}{2})Y(a) = -\rho_1(a)A(Y). \tag{2.8}$$

Putting $Y = \rho_1$ in (2.8), we obtain $\rho_1(a) = 0$. In this case, when $n > 3$, we get $Y(a) = 0$, for all Y. That is, a is a constant. Then, from (2.6), we get $\nabla S = 0$, which leads us the following result:

Theorem 2.2. *If the generator ρ_1 of $(HGQE)_n$, $(n > 3)$ is a parallel vector field, then the associated scalars of the manifold are constants and this manifold is Ricci symmetric.*

Moreover, since $S(X,Y) = g(QX,Y)$, for all X, Y where Q is a Ricci operator, from (1.11)-(1.13) and as $a \neq 0$, we can easily state that:

Corollary 2.3. *In a $(HGQE)_n$, the following statements hold:*

(1) $Q\rho_1$ is orthogonal to ρ_1 if and only if $a + b = 0$.

(2) $Q\rho_2$ is orthogonal to ρ_1, then $c = 0$. That is, the manifold becomes a generalized quasi Einstein manifold.

(3) $Q\rho_3$ is orthogonal to ρ_1, then $d = 0$. That is, the manifold becomes a generalized quasi Einstein manifold.

(4) $Q\rho_2$ can not be orthogonal to ρ_2 and ρ_3.

(5) $Q\rho_2$ is always orthogonal to ρ_3.

3 Some Pseudo-symmetry Types of $(HGQE)_n$

Definition 3.1. [8] An n-dimensional Riemannian manifold (M^n, g) is called Ricci-pseudosymmetric, if the tensor $R \cdot S$ and the Tachibana tensor $Q(g,S)$ are linearly dependent, where for all $X, Y, Z, W \in \chi(M)$;

$$(R(X,Y)\cdot S)(Z,W) = -S(R(X,Y)Z,W) - S(Z,R(X,Y)W), \tag{3.1}$$

$$Q(g,S)(Z,W;X,Y) = -S((X \wedge_g Y)Z,W) - S(Z,(X \wedge_g Y)W) \tag{3.2}$$

and

$$(X \wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{3.3}$$

That is, the necessary and sufficient condition for (M^n, g) to be a Ricci-pseudo symmetric manifold is that the following equation is satisfied

$$(R(X,Y)\cdot S)(Z,W) = L_S Q(g,S)(Z,W;X,Y) \tag{3.4}$$

on the set $U_S = \{x \in M : S \neq \frac{r}{n} g \ at\ x\}$ and L_S is a certain function on U_S. Then, by using (3.1)-(3.4), we can write

$$\begin{aligned} S(R(X,Y)Z,W) + S(Z,R(X,Y)W) = {} & L_S[g(Y,Z)S(X,W) \\ & -g(X,Z)S(Y,W) + g(Y,W)S(Z,X) - g(X,W)S(Y,Z)]. \end{aligned} \tag{3.5}$$

Now, we consider Ricci-pseudosymmetric $(HGQE)_n$. Using the fundamental equation (1.1) of $(HGQE)_n$ in (3.5) and in the resulting equation, putting $Z = \rho_1$, $W = \rho_2$ and $Z = \rho_1$, $W = \rho_3$ respectively, we obtain the following expressions:

$$bR(X,Y,\rho_1,\rho_2)+dR(X,Y,\rho_3,\rho_2) = L_S\Big[b[A(Y)B(X) - A(X)B(Y)] - dB(Y)D(X)\Big], \tag{3.6}$$

$$bR(X,Y,\rho_1,\rho_3)+cR(X,Y,\rho_2,\rho_3) = L_S\Big[b[A(Y)B(X) - A(X)B(Y)] + c[B(Y)D(X) - B(X)D(Y)] - D(X)D(Y)\Big]. \tag{3.7}$$

Contracting (3.7) over X and Y and remembering that the generators ρ_1, ρ_2 and ρ_3 are orthonormal, we obtain $d = 0$. Thus, the manifold under consideration reduces to a $G(QE)_n$. In [11], the authors proved that every Ricci-pseudosymmetric $G(QE)_n$ is an $N(k)$-quasi Einstein manifold. Hence, we can summarize above results by the following theorem:

Theorem 3.2. *Every Ricci-pseudosymmetric $(HGQE)_n$ is an $N(k)$-quasi Einstein manifold with $L_S = \frac{a+b}{n-1}$.*

If the function L_S in (3.4) vanishes, the Ricci-pseudosymmetric manifold turns into a Ricci semi-symmetric manifold. Thus, the next result is obtained directly:

Corollary 3.3. *Every Ricci semi-symmetric $(HGQE)_n$ is a quasi Einstein manifold whose Ricci tensor is of the form $S(X,Y) = a[g(X,Y) - A(X)A(Y)]$.*

As a generalization of Ricci-pseudosymmetric manifolds, R.Deszcz introduced the following notion:

Definition 3.4. [8] A semi-Riemannian manifold (M^n, g), $(n \geq 3)$ is said to be *Ricci-generalized pseudosymmetric* if at every point of (M^n, g), $R \cdot R$ and $Q(S,R)$ are linearly dependent.

That is, the necessary and sufficient condition for (M^n, g) to be a Ricci-generalized pseudosymmetric manifold is that the following equation is satisfied

$$R \cdot R = L_R Q(S,R) \tag{3.8}$$

at every point of the manifold, where L_R is some function on M.

Very important subclasses of Ricci-generalized pseudosymmetric manifolds form manifolds fulfilling the following condition

$$R \cdot R = Q(S,R). \tag{3.9}$$

Such manifolds are said to be *special Ricci-generalized pseudosymmetric manifolds*, [7].

The condition (3.9) arises during the study of the Riemannian manifolds satisfying the condition

$$\omega(X)R(Y,Z)+\omega(Y)R(Z,X)+\omega(Z)R(X,Y)=0, \tag{3.10}$$

where ω is a non-zero 1-form and $X, Y, Z \in \chi(M)$. R. Deszcz and W. Grycak obtained the following characterization theorem for this kind of manifold:

Theorem 3.5. *[9] (see Theorem 1) If at a point $x \in M$, the non-zero 1-form ω satisfies the condition* (3.10)*, then the relation* (3.9) *holds at $x \in M$.*

Motivated by the previous theorem, we will investigate the $(HGQE)_n$ satisfying the condition

$$\sum_{X,Y,Z} \omega(X)R(Y,Z)=0, \tag{3.11}$$

where $\sum$ denotes the cyclic sum over X, Y, Z. In this case, the following assumptions can be examined:

Case 1: First, we choose the 1-form ω as the associated 1-form A of $(HGQE)_n$. Then, we have

$$A(X)R(Y,Z)W+A(Y)R(Z,X)W+A(Z)R(X,Y)W=0. \tag{3.12}$$

Contracting (3.12) over Z and W, we get

$$A(X)S(Y,Z)+R(\rho_1,Y,X,Z)-A(Z)S(Y,X)=0. \tag{3.13}$$

Again, contracting (3.13) over X and Y, we obtain

$$2S(\rho_1,Z)-rA(Z)=0. \tag{3.14}$$

By virtue of (1.11) and (1.10), (3.14) yields

$$[(2-n)a+b]A(Z)+2cB(Z)+2dB(Z)=0. \tag{3.15}$$

Putting $Z=\rho_1$ in (3.15), we get $b=(n-2)a$, putting $Z=\rho_2$ in (3.15), we get $c=0$ and putting $Z=\rho_3$ in (3.15), we get $d=0$. Hence the Ricci tensor can be expressed in the following form:

$$S(X,Y)=a[g(X,Y)+(n-2)A(X)A(Y)]. \tag{3.16}$$

Since $n>2$, the manifold reduces to a $(QE)_n$. Also, in view of (3.16), (3.13) yields

$$R(X,Y)\rho_1=a[A(Y)X-A(X)Y] \tag{3.17}$$

which implies that the generator ρ_1 belongs to a-nullity distribution. Hence, such a manifold becomes an $N(k)$-quasi Einstein manifold with $k=\frac{a+b}{n-1}=a$.

Case 2-3: If we choose the 1-form ω as the associated 1-form B or D of $(HGQE)_n$, then from similar calculations with the Case 1, we get $c = 0$ and $b = a(2-n)$. Thus, the Ricci tensor can be expressed as

$$S(X,Y) = ag(X,Y) + a(2-n)A(X)A(Y) + d[A(X)\omega(Y) + A(Y)\omega(X)] \tag{3.18}$$

which means that the manifold reduces to a $G(QE)_n$.

As a result of these examinations, we can state the following theorem:

Theorem 3.6. *Let M be a $(HGQE)_n$ $(n > 2)$ satisfying the condition*

$$\sum_{X,Y,Z} \omega(X)R(Y,Z) = 0,$$

where ω is a certain 1-form and $\sum$ denotes the cyclic sum over $X, Y, Z \in \chi(M)$. Then, the following conditions hold:

(1) If $\omega = A$, then M reduces to an $N(a)$-quasi Einstein manifold, where a is an associated scalar function of M.

(2) If $\omega = B$ or D, then M reduces to a $G(QE)_n$.

Also, as a result of the last two theorems, the following corollary is obtained:

Corollary 3.7. *Every $(HGQE)_n$ $(n > 2)$ satisfying the condition*

$$\sum_{X,Y,Z} \omega(X)R(Y,Z) = 0,$$

where ω is one of the associated 1-forms of the manifold, is a special Ricci-generalized pseudosymmetric manifold.

Analogously, we can examine the $(HGQE)_n$ satisfying the condition

$$\sum_{X,Y,Z} \omega(X)C(Y,Z) = 0, \tag{3.19}$$

where ω is a non-zero 1-form, $\sum$ denotes the cyclic sum over X, Y, Z and C denotes the conformal curvature tensor defined by

$$\begin{aligned} C(X,Y)Z =& R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{3.20}$$

where Q is Ricci operator, r is scalar curvature tensor. Similarly, the following cases can be investigated:

Case 1: We can choose the 1-form ω as the associated 1-form A of the $(HGQE)_n$. Then, we have

$$A(X)C(Y,Z)W + A(Y)C(Z,X)W + A(Z)C(X,Y)W = 0. \tag{3.21}$$

Then, contracting (3.21), first we obtain $C(X,Y)Z = 0$ for all X, Y, Z. That is, the manifold is conformally flat. Thus from (3.20), we get

$$\begin{aligned} R(X,Y)Z =& \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &- \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{3.22}$$

Putting $Z = \rho_1$ in (3.22), we get

$$R(X,Y)\rho_1 = \frac{a+b}{n-1}[A(Y)X - A(X)Y] - \frac{d}{n-2}[D(Y)X - D(X)Y] \tag{3.23}$$

and contracting (3.23) over X, we get

$$S(X,\rho_1) = (a+b)A(X) + \frac{d(n-1)}{n-2}D(X). \tag{3.24}$$

Comparing the equations (1.11) and (3.24) and putting $X = \rho_2$ and $X = \rho_3$ in the resulting equation, we get $c = 0$ and $d = 0$, respectively. Similarly, putting $Z = \rho_2$ in (3.22), we get

$$\begin{aligned} R(X,Y)\rho_2 =& \Big(\frac{2a}{n-2} - \frac{r}{(n-1)(n-2)}\Big)[B(Y)X - B(X)Y] \\ &+ \frac{c-d}{n-2}[A(Y)X - A(X)Y] \end{aligned} \tag{3.25}$$

and contracting (3.25) over X, we get

$$S(X,\rho_2) = \frac{2a(n-1) - r}{(n-2)}B(X) + \frac{(c-d)(n-1)}{n-2}A(X). \tag{3.26}$$

Comparing the equations (1.12) and (3.26) and putting $X = \rho_2$ in the resulting equation, we also have $b = 0$. In summary, we have $b = c = d = 0$. This implies that the manifold under consideration reduces to an Einstein manifold, which is a contradiction.

Case 2-3: If we choose the 1-form ω as the associated 1-form B or D of the $(HGQE)_n$, then by direct calculations, we obtain a contradiction similar with the Case 1. Hence, we obtain the following result:

Theorem 3.8. *There does not exist any $(HGQE)_n$ satisfying the condition*

$$\sum_{X,Y,Z} \omega(X)C(Y,Z) = 0,$$

where ω is one of the associated 1-forms of the manifold, $\sum$ denotes the cyclic sum over X, Y, Z and C denotes the conformal curvature tensor.

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