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Quarter Symmetric Connections On Complex Weyl Manifolds

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Abstract. In this work, we study quarter symmetric linear connections on Kähler Weyl manifolds and almost contact Weyl manifolds.
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1 Introduction

A gauge invariant theory which unifies the gravity and the electromagnetic fields was first introduced by Weyl in 1918 [13]. For physical reasons, his theory was not accepted but it remained both as a part of physics and mathematics. Weyl manifold is a differentiable manifold with a torsion free connection which is non-metric.

In 1924, Friedmann and Schouten introduced a semi-symmetric linear connection in a differentiable manifold [4]. After that, in 1932, Hayden introduced the notion of metric connection with torsion in a Riemannian manifold [6]. Moreover, Yano studied semi-symmetric metric connection in a Riemannian manifold and obtained a result about conformally flatness [14].

The notion of semi symmetric connection was generalized to quarter symmetric connection by Golab in 1975 [5]. There are many papers about quarter symmetric connection not only in Riemannian manifolds but also in Hermitian, Kähler, Kenmotsu manifolds (see Mishra and Pandey [9], Dwivedi [3], Pusic [12], Yano and Imai [15]).

In this work, we consider a quarter symmetric connection on Kähler Weyl manifolds and almost contact Weyl manifolds and examine the properties of this connection.

2 Preliminaries

A Weyl manifold is a differentiable manifold M of dimension n with a conformal metric tensor g and a symmetric connection D which satisfies, called the compatibility condition,

$$D_k g_{ij} = 2\omega_k g_{ij}, \tag{2.1}$$

where ω is a 1-form. Such a Weyl manifold is denoted by $M_n(g,\omega)$. If ω is a closed form, $M_n(g,\omega)$ is conformal to a Riemannian manifold.

Under the conformal change of the metric tensor g,

$$\breve{g}_{ij} = \lambda^2 g_{ij}, \quad \lambda > 0, \tag{2.2}$$

the 1-form ω changes by the law

$$\breve{\omega}_k = \omega_k + D_k \ln \lambda. \tag{2.3}$$

A quantity S is called a satellite of g with weight r if it admits a transformation of the form

$$\breve{S} = \lambda^r S, \tag{2.4}$$

under the change (2.2) of the metric tensor g.

The prolonged (extended) covariant derivative of a satellite S of weight r is defined by

$$\dot{D}_k S = D_k S - r w_k S, \tag{2.5}$$

from which it follows that $\dot{D}_k g_{ij} = 0$ (see [16], [7], [10]).

It is easy to see from (2.1) that

$$\Gamma^i_{kl} = \left\{ \begin{matrix} i \\ kl \end{matrix} \right\} - g^{im}(g_{mk}\omega_l + g_{ml}\omega_k - g_{kl}\omega_m), \tag{2.6}$$

where Γ^i_{jk} are the coefficients of the Weyl connection D and $\left\{ \begin{smallmatrix} i \\ kl \end{smallmatrix} \right\}$ are the connection coefficients of the Levi-Civita connection.

The mixed curvature tensor, the covariant curvature tensor, the Ricci tensor and the scalar curvature for $M_n(g,\omega)$ are respectively given by [11]:

$$\begin{aligned} v^j W^p_{jkl} &= (D_k D_l - D_l D_k) v^p, & (2.7)\\ W_{hjkl} &= g_{hp} W^p_{jkl}, & (2.8)\\ W_{ij} &= W^p_{ijp} = g^{hk} W_{hijk}, & (2.9)\\ s &= g^{ij} W_{ij}. & (2.10) \end{aligned}$$

By considering (2.7), the explicit form of the mixed curvature tensor W^p_{jkl} for $M_n(g,\omega)$ is

$$W^p_{jkl} = \partial_k \Gamma^p_{jl} - \partial_l \Gamma^p_{jk} + \Gamma^p_{hk}\Gamma^h_{jl} - \Gamma^p_{hl}\Gamma^h_{jk}. \tag{2.11}$$

The mixed curvature tensor, the covariant curvature tensor and the Ricci tensor of $M_n(g,\omega)$ satisfy the following properties [11]:

$$\begin{aligned} W_{ijkl} + W_{ijlk} &= 0, \quad W^i_{ikl} = n(D_l\omega_k - D_k\omega_l) = 2nD_{[l}\omega_{k]}, & (2.12)\\ W_{ijkl} + W_{jikl} &= 4g_{ij} D_{[l}\omega_{k]}, \quad W_{[ij]} = nD_{[i}\omega_{j]}. & (2.13) \end{aligned}$$

Let N be an n-dimensional Riemannian manifold endowed with a linear connection ∇. Then ∇ is said to be quarter symmetric if the torsion tensor T^i_{jk} of ∇ satisfies

$$T^i_{jk} = p_j A^i_k - p_k A^i_j, \tag{2.14}$$

where p_k is a 1-form and A^j_i is any $(1,1)$ tensor field. If A_{jk} is defined as $A^i_j g_{jk} = A_{jk}$, then

$$A_{jk} = U_{jk} + V_{jk}, \tag{2.15}$$

where U_{jk} and V_{jk} are respectively symmetric and anti symmetric part of A_{jk} [15].

3 Kähler Weyl Manifolds

A Weyl manifold of dimension $2n$ is called Kähler Weyl if

$$F^i_j F^h_i = -\delta^h_j, \tag{3.1}$$

$$F^t_j F^s_i g_{ts} = g_{ji}, \tag{3.2}$$

and

$$\dot{D}_j F^k_i = 0, \quad \forall i, j, k, \tag{3.3}$$

where F^j_i is a $(1,1)$ tensor field of weight zero and called an almost complex structure, g_{ij} is Hermitian metric. Such a manifold will be denoted by $KM_{2n}(g,\omega)$ [2].

The $(0,2)$ tensor field F_{ij} of weight 2 and $(2,0)$ tensor field F^{ij} of weight -2 are, respectively, given by

$$F_{ij} = F^k_i g_{kj} = -F_{ji} \tag{3.4}$$

and

$$F^{ij} = F^j_k g^{ik} = -F^{ji}. \tag{3.5}$$

The contraction of (3.4) on the indices k and i gives $F^i_i = 0$.

Suppose that a Kähler Weyl manifold admits a quarter symmetric linear connection $\bar{D}$ with the torsion tensor $\bar{T}$ and satisfies the following compatibility condition

$$\bar{D}_k g_{ij} = 2\omega_k g_{ij}. \tag{3.6}$$

If we take $U_{jk} = g_{jk}$ and $V_{jk} = F_{jk}$ in (2.15), then we find that $A^k_i = \delta^k_i + F^k_i$. Therefore, the torsion tensor $\bar{T}$ takes the form

$$\bar{T}^i_{jk} = p_j \left(\delta^i_k + F^i_k\right) - p_k \left(\delta^i_j + F^i_j\right). \tag{3.7}$$

We note that the 1-form p_k is of zero weight.

Theorem 3.1. *On every Kähler Weyl manifold there exists a unique quarter symmetric linear connection associated to every 1-form p and $(1,1)$ tensor field F.*

Proof. Assume that the relation between $\bar{\Gamma}^j_{mk}$ and Γ^j_{mk} is given by

$$\bar{\Gamma}^j_{mk} = \Gamma^j_{mk} + U^j_{mk}, \tag{3.8}$$

where $\bar{\Gamma}^j_{mk}$ and Γ^j_{mk} are respectively the coefficients of $\bar{D}$ and D, and U^j_{mk} is any $(1,2)$ tensor field. Then from (2.1), (3.6) and (3.8), we find that

$$U^h_{ik}g_{hj} + U^h_{jk}g_{hi} = 0. \tag{3.9}$$

After permuting the indices i, j and k in the above equation cyclicly and using some algebraic operations, we obtain

$$\begin{aligned}(U^h_{ik} + U^h_{ki})g_{hj} &= (U^h_{ij} - U^h_{ji})g_{hk} + (U^h_{kj} - U^h_{jk})g_{hi} \\ &= \bar{T}^h_{ij}g_{hk} + \bar{T}^h_{kj}g_{hi}. \end{aligned} \tag{3.10}$$

From (3.7) and after some simplifications, we have

$$U^t_{ik} + U^t_{ki} = p_i\left(\delta^t_k - F^t_k\right) + p_k\left(\delta^t_i - F^t_i\right) - 2p^t g_{ik}, \tag{3.11}$$

where $p^t = p_k g^{tk}$. Since $\bar{T}^t_{ik} = U^t_{ik} - U^t_{ki}$, by considering the above equation, we obtain

$$U^t_{ik} = p_i\delta^t_k - p_k F^t_i - p^t g_{ik}. \tag{3.12}$$

Hence, we find that $\bar{\Gamma}^j_{mk} = \Gamma^j_{mk} + p_m\delta^j_k - p_k F^j_m - p^j g_{mk}$ which completes the proof. □

The mixed curvature tensor $\bar{W}^i_{jkl}$ for $\bar{D}$ is given by

$$\bar{W}^i_{jkl} = \partial_k\bar{\Gamma}^i_{jl} - \partial_l\bar{\Gamma}^i_{jk} + \bar{\Gamma}^m_{jl}\bar{\Gamma}^i_{mk} - \bar{\Gamma}^m_{jk}\bar{\Gamma}^i_{mk}. \tag{3.13}$$

Hence, by considering the definition of $\bar{\Gamma}^i_{jl}$ and after a long straightforward calculations, we obtain

$$\begin{aligned}\bar{W}^i_{jkl} &= W^i_{jkl} + \delta^i_l\alpha_{jk} - \delta^i_k\alpha_{jl} + g_{jk}g^{it}\alpha_{tl} - g_{jl}g^{it}\alpha_{tk} - 2F^i_j\dot{D}_{[k}p_{l]} \\ &+ p^i\left(F_{jk}p_l - F_{jl}p_k\right) + p_j\left(F^i_k p_l - F^i_l p_k\right), \end{aligned} \tag{3.14}$$

where $\alpha_{jk} = \dot{D}_k p_j - p_j p_k + F^m_j p_m p_k + \frac{1}{2}g_{jk}p^m p_m$.

Therefore, the covariant curvature tensor $\bar{W}_{ijkl}$, the Ricci tensor $\bar{W}_{jk}$ and the scalar curvature $\bar{s}$ are respectively given by

$$\begin{aligned}\bar{W}_{ijkl} &= W_{ijkl} + g_{il}\alpha_{jk} - g_{ik}\alpha_{jl} + g_{jk}\alpha_{il} - g_{jl}\alpha_{ik} + 2F_{ij}\dot{D}_{[k}p_{l]} \\ &+ p_i\left(F_{jk}p_l - F_{jl}p_k\right) + p_j\left(F_{il}p_k - F_{ik}p_l\right), \end{aligned} \tag{3.15}$$

$$\begin{aligned}\bar{W}_{jk} &= W_{jk} + (n-2)\alpha_{jk} + g_{jk}g^{il}\alpha_{il} - g_{jl}\alpha_{ik} + 2g^{il}F_{ij}\dot{D}_{[k}p_{l]} \\ &+ F_{jk}p_l p^l - F_{jl}p^l p_k - F_{lk}p^l p_j, \end{aligned} \tag{3.16}$$

and

$$\bar{s} = s + 2(n-1)g^{jk}\alpha_{jk} + 2F^{lk}\dot{D}_{[k}p_{l]}. \tag{3.17}$$

Proposition 3.2. *The mixed curvature tensor $\bar{W}^i_{ikl}$ and the covariant curvature tensor $\bar{W}_{ijkl}$ of a Kähler Weyl manifold with a quarter symmetric linear connection satisfy the following relations:*

(i) $\bar{W}_{ijkl} + \bar{W}_{ijlk} = 0$

(ii) $\bar{W}_{ijkl} + \bar{W}_{jikl} = 4g_{ij}D_{[l}\omega_{k]}$

(iii) $\bar{W}^i_{ikl} = W^i_{ikl} = 2nD_{[l}\omega_{k]}$.

Proof. (i) The covariant curvature tensor $\bar{W}_{ijkl}$ of a Kähler Weyl manifold endowed with a quarter symmetric linear connection $\bar{D}$ is given by (3.15). By changing the indices k and l and then taking the sum of the equations obtained gives $\bar{W}_{ijkl} + \bar{W}_{ijlk} = W_{ijkl} + W_{ijlk} = 0$.

(ii) Similarly, if we change the indices k and l in (3.15) and sum up the obtained equations, then we get $\bar{W}_{ijkl} + \bar{W}_{jikl} = W_{ijkl} + W_{jikl} = 4g_{ij}D_{[l}\omega_{k]}$.

(iii) Since $F^i_i = 0$, the result follows easily from (3.14). □

Theorem 3.3. *If the curvature tensor of a Kähler Weyl manifold with a quarter symmetric linear connection vanishes and the* 1*-form p_k is locally a gradient, then the connection reduces to the Weyl connection.*

Proof. If $\bar{W}_{ijkl} = 0$ and the 1-form p_k is locally a gradient, then (3.15) takes the form

$$\begin{aligned} W_{ijkl} &= -g_{il}\alpha_{jk} + g_{ik}\alpha_{jl} - g_{jk}\alpha_{il} + g_{jl}\alpha_{ik} \\ &- p_i\left(F_{jk}p_l - F_{jl}p_k\right) - p_j\left(F_{il}p_k - F_{ik}p_l\right). \end{aligned} \tag{3.18}$$

If we permute the indices j, k and l in (3.18) cyclicly, then we obtain two more equations. Now, by taking the sum of the three equations and taking into account of the 1^{st} Bianchi Identity for Weyl manifolds, we obtain

$$0 = g_{il}\alpha_{[jk]} + g_{ij}\alpha_{[kl]} + g_{ik}\alpha_{[lj]} + F_{jk}p_ip_l + F_{kl}p_ip_j + F_{lj}p_ip_k. \tag{3.19}$$

By contracting the above equation with $F^{jl}g^{ik}$, we get for $n \neq 2$

$$F^{jl}\alpha_{[jl]} = -p^kp_k. \tag{3.20}$$

Since $\alpha_{[jl]} = \dfrac{1}{2}\left(F_{jm}p^mp_l - F_{lm}p^mp_j\right)$,

$$\begin{aligned} -p^kp_k &= \frac{1}{2}F^{jl}\left(F_{jm}p^mp_l - F_{lm}p^mp_j\right) \\ &= \frac{1}{2}\left(\delta^l_mp^mp_l + \delta^j_mp^mp_j\right) \\ &= p^kp_k, \end{aligned} \tag{3.21}$$

from which we find that $p_k = 0$ for positive definite metric tensors belonging to the conformal class. Now since $p_k = 0$, $\bar{\Gamma}^j_{mk} = \Gamma^j_{mk} + p_m\delta^j_k - p_kF^j_m - p^jg_{mk}$ takes the form $\bar{\Gamma}^j_{mk} = \Gamma^j_{mk}$ which completes the proof. □

4 Almost Contact Weyl Manifolds

Let M_{2n+1} be a differentiable manifold of dimension $2n+1$. An almost contact structure (ϕ, ξ, η) on M_{2n+1} is a triple satisfying the following relations

$$\phi_i^j \phi_j^k = -\delta_i^k + \eta_i \xi^k, \tag{4.1}$$

$$\eta_i \xi^i = 1, \tag{4.2}$$

$$\phi_i^j \xi^i = 0, \tag{4.3}$$

$$\eta_i \phi_j^i = 0, \tag{4.4}$$

where ϕ_i^j is a tensor field of type $(1,1)$, ξ^i is a vector field and η_i is a 1-form. Moreover, if there is given a Riemannian metric g_{ij} such that

$$g_{ij}\phi_t^i \phi_s^j = g_{ts} - \eta_t \eta_s, \tag{4.5}$$

$$g_{ij}\xi^j = \eta_i, \tag{4.6}$$

then (ϕ, ξ, η, g) is called an almost contact metric structure on M. A differentiable manifold M_{2n+1} with almost contact metric structure (ϕ, ξ, η, g) is called almost contact metric manifold [1].

It is easy to see that the tensor ϕ_{ij}, which is defined by $\phi_i^k g_{jk} = \phi_{ij}$, is anti symmetric and contraction of ϕ_i^j gives $\phi_i^i = 0$.

It follows immediately from the equations (4.1), (4.2) and (4.5) that the $(1,1)$ tensor field ϕ_i^j , the 1-form η_i and the vector field ξ^i are weight of 0, 1 and -1, respectively.

Let $M_{2n+1}(g,\omega)$ be a Weyl manifold with the connection D. Then $M_{2n+1}(g,\omega)$ has an almost contact structure if the following conditions are satisfied in addition to the conditions (4.1)-(4.6) [8]:

$$\dot{D}_k g_{ij} = 0, \quad \dot{D}_k \phi_i^j = 0, \quad \dot{D}_k \eta_i = 0, \quad \dot{D}_k \xi^i = 0. \tag{4.7}$$

Such a manifold is called almost contact Weyl manifold and will be denoted by $ACM_{2n+1}(g,\omega)$.

Now, we consider the manifold $ACM_{2n+1}(g,\omega)$ with a quarter symmetric linear connection $\widetilde{D}$ and the torsion tensor $\widetilde{T}$ is of the form

$$\widetilde{T}_{jk}^i = q_j \phi_k^i - q_k \phi_j^i, \tag{4.8}$$

where the $(1,1)$ tensor field $A_i^j = \phi_i^j$ and the 1-form $q_j = f\eta_j$, where f is any function of weight -1. Here, we also have

$$\widetilde{D}_k g_{ij} = 2\omega_k g_{ij}. \tag{4.9}$$

Theorem 4.1. *On every almost contact Weyl manifold there exists a unique quarter symmetric linear connection associated to every 1-form q and $(1,1)$ tensor field ϕ.*

Proof. Suppose that the relation between $\widetilde{\Gamma}^j_{mk}$ and Γ^j_{mk} be

$$\widetilde{\Gamma}^j_{mk} = \Gamma^j_{mk} + U^j_{mk}, \tag{4.10}$$

where $\widetilde{\Gamma}^j_{mk}$ and Γ^j_{mk} are the coefficients of the connections $\widetilde{D}$ and D, respectively and U^j_{mk} is any tensor field of type $(1,2)$.

From (2.1), (4.9) and (4.10) we have

$$U^h_{ik} g_{hj} + U^h_{jk} g_{hi} = 0. \tag{4.11}$$

Permuting cyclicly the indices i, j, k and after some modifications, we obtain

$$\begin{aligned} (U^h_{ik} + U^h_{ki}) g_{hj} &= (U^h_{ij} - U^h_{ji}) g_{hk} + (U^h_{kj} - U^h_{jk}) g_{hi} \\ &= \widetilde{T}^h_{ij} g_{hk} + \widetilde{T}^h_{kj} g_{hi} \end{aligned} \tag{4.12}$$

Using the definition of $\widetilde{T}^h_{ij}$ and after straightforward calculations, we find that

$$(U^h_{ik} + U^h_{ki}) g_{hj} = -q_i \phi_{kj} - q_k \phi_{ij}. \tag{4.13}$$

Multiplying the last equation by g^{jt}, we have

$$U^t_{ik} + U^t_{ki} = -q_i \phi^t_k - q_k \phi^t_i. \tag{4.14}$$

We conclude from the above equation that

$$\widetilde{T}^h_{ij} = U^t_{ik} - U^t_{ki} = q_i \phi^t_k - q_k \phi^t_i. \tag{4.15}$$

Hence, we obtain

$$U^t_{ik} = -q_k \phi^t_i, \tag{4.16}$$

and therefore

$$\widetilde{\Gamma}^j_{mk} = \Gamma^j_{mk} - q_k \phi^j_m. \tag{4.17}$$

□

The mixed curvature tensor for an almost contact Weyl manifold endowed with a quarter symmetric linear connection $\widetilde{D}$ is of the form

$$\widetilde{W}^i_{jkl} = \partial_k \widetilde{\Gamma}^i_{jl} - \partial_l \widetilde{\Gamma}^i_{jk} + \widetilde{\Gamma}^m_{jl} \widetilde{\Gamma}^i_{mk} - \widetilde{\Gamma}^m_{jk} \widetilde{\Gamma}^i_{mk}. \tag{4.18}$$

By using (4.17) and (4.18) we have

$$\begin{aligned} \widetilde{W}^i_{jkl} &= \partial_k \left(\Gamma^i_{jl} - q_l \phi^i_j\right) - \partial_l \left(\Gamma^i_{jk} - q_k \phi^i_k\right) \\ &+ \left(\Gamma^m_{jl} - q_l \phi^m_j\right) \left(\Gamma^i_{mk} - q_k \phi^i_m\right) - \left(\Gamma^m_{jk} - q_k \phi^m_j\right) \left(\Gamma^i_{ml} - q_l \phi^i_m\right) \\ &= W^i_{jkl} - \partial_k \left(q_l \phi^i_j\right) + \partial_l \left(q_k \phi^i_j\right) - q_k \Gamma^m_{jl} \phi^i_m - q_l \Gamma^i_{mk} \phi^m_j \\ &+ q_l q_k \phi^m_j \phi^i_m + q_l \Gamma^m_{jk} \phi^i_m + q_k \Gamma^i_{ml} \phi^m_j - q_k q_l \phi^m_j \phi^i_m \end{aligned} \tag{4.19}$$

After some simplifications we obtain

$$\begin{aligned}\widetilde{W}^i_{jkl} &= W^i_{jkl} + \phi^i_j\left(\partial_l q_k - \partial_k q_l\right) + q_k\left(\partial_l \phi^i_j - \Gamma^m_{jl}\phi^i_m + \Gamma^i_{ml}\phi^m_j\right) \\ &- q_l\left(\partial_k \phi^i_j - \Gamma^m_{jk}\phi^i_m + \Gamma^i_{mk}\phi^m_j\right). \end{aligned} \tag{4.20}$$

If we use the definition of prolonged covariant derivative for ϕ^i_j and η_k, then we get

$$\widetilde{W}^i_{jkl} = W^i_{jkl} + \phi^i_j\left(\dot{D}_l q_k - \dot{D}_k q_l\right) + q_k \dot{D}_l \phi^i_j - q_l \dot{D}_k \phi^i_j. \tag{4.21}$$

Since $\dot{D}_l \phi^i_j = 0$, we find that

$$\widetilde{W}^i_{jkl} = W^i_{jkl} + \phi^i_j\left(\dot{D}_l q_k - \dot{D}_k q_l\right). \tag{4.22}$$

From (4.22), the covariant curvature tensor $\widetilde{W}_{ijkl}$ is given by

$$\widetilde{W}_{ijkl} = W_{ijkl} + 2\phi_{ji}\dot{D}_{[l}q_{k]}, \tag{4.23}$$

where $\dot{D}_{[l}q_{k]}$ is anti symmetric part of $\dot{D}_l q_k$.

Contracting the tensor $\widetilde{W}^i_{jkl}$ with respect to i and l and using the fact that $\eta_i\phi^i_j = 0$, gives us

$$\widetilde{W}_{jk} = W_{jk} + \phi^i_j \dot{D}_i q_k. \tag{4.24}$$

Theorem 4.2. *For an almost contact Weyl manifold with a quarter symmetric linear connection, we have*

$$\widetilde{s} = s, \tag{4.25}$$

where $\widetilde{s}$ and s are scalar curvature of the manifold with respect to the connections $\widetilde{D}$ and D, respectively.

Proof. Multiplying (4.24) by g^{jk} and using the identity $\xi^j\phi^i_j = 0$ gives

$$\begin{aligned}\widetilde{s} &= s + \dot{D}_i\left(q_k g^{jk}\phi^i_j\right) \\ &= s + \dot{D}_i\left(f\eta_k g^{jk}\phi^i_j\right) \\ &= s + \dot{D}_i\left(f\xi^j\phi^i_j\right) \\ &= s. \end{aligned}$$

□

Theorem 4.3. *On an almost contact Weyl manifold with a quarter symmetric linear connection, if the 1-form q is locally a gradient, then*

$$\begin{aligned}\widetilde{W}^i_{jkl} &= W^i_{jkl}, \\ \widetilde{W}_{jk} &= W_{jk}. \end{aligned}$$

Proof. The proof is immediate from (4.22). □

Proposition 4.4. *On an almost contact Weyl manifold with a quarter symmetric connection, the following relations hold:*

(i) $\widetilde{W}_{ijkl} + \widetilde{W}_{ijlk} = 0$

(ii) $\widetilde{W}_{ijkl} + \widetilde{W}_{jikl} = \dfrac{4}{n} g_{ij} W_{[lk]}$

(iii) $\widetilde{W}^i_{ikl} = 2nD_{[l}\omega_{k]}$

(iv) $\widetilde{W}_{klij} + \widetilde{W}_{kijl} + \widetilde{W}_{kjli} = 2\left(\phi_{kl}\dot{D}_{[i}q_{j]} + \phi_{ki}\dot{D}_{[j}q_{l]} + \phi_{kj}\dot{D}_{[l}q_{i]}\right)$

Proof. (i). Changing the indices k and l in (4.23) yields

$$\widetilde{W}_{ijlk} = W_{ijlk} - 2\phi_{ji}\dot{\nabla}_{[l}q_{k]}. \tag{4.26}$$

By adding (4.23) to (4.26) we obtain the result.

(ii). Similar to proof (i).

(iii). Contracting (4.22) with respect to i and j yields

$$\widetilde{W}^i_{ikl} = W^i_{ikl} + \phi^i_i\left(\dot{D}_l q_k - \dot{D}_k q_l\right). \tag{4.27}$$

Since $\phi^i_i = 0$, we find that

$$\widetilde{W}^i_{ikl} = W^i_{ikl} = 2nD_{[l}\omega_{k]}. \tag{4.28}$$

(iv). Using the 1^{st} Bianchi Identity

$$W_{klij} + W_{kijl} + W_{kjli} = 0, \tag{4.29}$$

and after the straightforward calculations, we get the result. □

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