Proceedings Book of International Workshop on Theory of Submanifolds (Volume: 1 (2016)) June 2–4, 2016, Istanbul, Turkey. Editors: Nurettin Cenk Turgay, Elif Özkara Canfes, Joeri Van der Veken and Cornelia-Livia Bejan
Recieved: February 03, 2017
Accepted: March 02, 2017
DOI: 10.24064/iwts2016.2017.3

On Some Geometric Structures on the Cotangent Bundle of a Manifold

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Abstract. Let (M, ∇) be a manifold with a symmetric linear connection. The natural Riemann extension $\overline{g}$ (defined by Kowalski-Sekizawa) generalizes the Riemann extension (introduced by Patterson-Walker). The harmonic morphisms form a special class of harmonic maps, with many applications [1]. On $(T^*M, \overline{g})$ we obtain here a para-Hermitian structure, we construct a harmonic morphism and we generalize a result of [2].

Keywords. Cotangent bundle · Harmonicity · Riemann extension.
MSC 2010 Classification. Primary: 53C15; Secondary:53C43.

Dedicated to the memory of Professor Ioan Gottlieb and to his wife, Professor Cleopatra Mociutchi

1 Introduction

The affine geometry of a manifold (M, ∇), endowed with a symmetric linear connection, induces a (semi-)Riemannian geometry on the total space of the cotangent bundle T^*M, given by the Riemann extension $\overline{g}$ introduced by Patterson-Walker [8].

Osserman problem, Walker manifolds, almost para-Hermitian manifolds, non-Lorentzian geometry and so on, are related to the Riemann extension. For some other applications, see [5].

The Riemann extension is a metric of signature (n, n) on T^*M which was generalized by Kowalski and Sekizawa to the natural Riemann extension (see [7], [9] and for the notion of naturality see [6]). Another generalization is the deformed Riemann extension (see [4]).

In [2], the harmonic functions were characterized with respect to both natural Riemann extension and (classical) Riemann extension on the phase space T^*M.

A special class of harmonic maps is given by harmonic morphisms, see [1]. A harmonic morphism between (semi-)Riemannian manifolds is defined as a smooth map between (semi-)Riemannian manifolds which pulls back (local) harmonic functions from the target manifold to (local) harmonic functions on the domain manifold.

The present paper gives some applications of the deformed and natural Riemann extensions on T^*M. Our main goal is to provide some harmonic morphisms with respect to natural Riemann extension on T^*M.

Note: This paper is an announcement of the forthcoming paper [3]. The geometric structures, induced from the base manifold M to the total space of its cotangent bundle T^*M, can also be seen as some extensions of several geometric objects from a submanifold to the whole space.

2 Preliminaries

The technique of lifting several geometric objects from a base n-dimensional manifold M to its cotangent bundle T^*M goes back to the last century, for which we cite [10]. The natural projection $p : T^*M \to M$, $p(x,w) = x$ associates to each local chart $(U, x^1, \ldots, x^n)$ around $x \in M$ the corresponding local chart $(p^{-1}(\mathcal{U}), x^1, \ldots, x^n, x^{1*}, \ldots, x^{n*})$ around $(x,w) \in T^*M$. On the cotangent space $(T^*M)_{(x,w)}$ at (x,w) of T^*M one has a canonical basis:

$$\{(\partial_1)_{(x,w)}, \ldots, (\partial_n)_{(x,w)}, (\partial_{1*})_{(x,w)}, \ldots, (\partial_{n*})_{(x,w)}\},$$

where $\partial_i = \partial/\partial x^i$ and $\partial_{i*} = \partial/\partial w_i$, $i = \overline{1,n}$. In local coordinates, the global defined vertical vector field:

$$W = \sum_{i=1}^{n} w_i \partial_{i*}$$

is of Liouville type.

The vertical lift $f^v \in \mathcal{F}(T^*M)$ of any function $f \in \mathcal{F}(M)$ is defined by $f^v = f \circ p$. Then the vertical lift X^v is a function on T^*M associated to the vector field $X \in \mathcal{X}(M)$ and defined by:

$$X^v(x,w) = w(X_x)$$

at any point $(x,w) \in T^*M$. When X is written in local coordinates as $X = \sum_{i=1}^{n} \xi^i \partial_i$ then X^v can be written in local coordinates as $X^v(x,w) = \sum_{i=1}^{n} w_i \xi^i(x)$ at any point $(x,w) \in T^*M$.

We note that X^v is not a vector field on T^*M but X^v is a function preserved by the action of the canonical vector field W, that is:

$$W(X^v) = X^v, \ \ \forall X \in \mathcal{X}(M). \tag{2.1}$$

Proposition 2.1. *([11]) If X and Y are vector fields on T^*M such that $X(Z^v) = Y(Z^v)$, $\forall Z \in \mathcal{X}(M)$ then $X = Y$.*

We recall from [10] that the vertical lift α^v is a vector field tangent to T^*M, associated to any 1-form $\alpha \in \Omega^1(M)$ and defined by $\alpha^v(Z^v) = (\alpha(Z))^v$, $\forall Z \in \mathcal{X}(M)$. When α is written in local coordinates as $\alpha = \sum_{i=1}^n \alpha_i dx_i$ then α^v can be written in local coordinates as $\alpha^v = \sum_{i=1}^n \alpha_i \partial_{i*}$ where we identify $f^v = f \circ p \in \mathcal{F}(T^*M)$ for any $f \in \mathcal{F}(M)$. Hence $\alpha^v(f^v) = 0$, $\forall f \in \mathcal{F}(M)$. The complete lift of a vector field $X \in \mathcal{X}(M)$ is a vector field $X^c \in \mathcal{X}(T^*M)$ defined by:

$$X^c(Z^v) = [X, Z]^v, \ \ \forall Z \in \mathcal{X}(M).$$

When X is written in local coordinates by $X = \sum_{i=1}^n \xi^i \partial_i$ then X^c can be written in local coordinates as:

$$X^c_{(x,w)} = \sum_{i=1}^n \xi^i(x)(\partial_i)_{(x,w)} - \sum_{h,i=1}^n w_h(\partial_i \xi^h)(x)(\partial_{i*})_{(x,w)},$$

at each point $(x, w) \in T^*M$.

Then $X^c(f^v) = (Xf)^v$, $\forall f \in \mathcal{F}(M)$ and on T^*M the Lie bracket satisfies:

$$[X^c, Y^c] = [X, Y]^c, \ \ [X^c, \alpha^v] = (\mathcal{L}_X \alpha)^v,$$
$$[\alpha^v, \beta^v] = 0 = [X^c, W], [\alpha^v, W] = \alpha^v, \forall X, Y \in \mathcal{X}(M), \alpha, \beta \in \Omega^1(M),$$

where $\mathcal{L}_X$ denotes the Lie derivative with respect to X.

3 Deformed Riemann extension

If (M, ∇) is an n-dimensional manifold endowed with a symmetric linear connection, then the deformed Riemann extension is a semi-Riemannian metric $\overline{g}$ of signature (n, n) on the total space of T^*M defined at any $(x, w) \in T^*M$ by

$$\overline{g}_{(x,w)}(X^c, Y^c) = -aw(\nabla_X Y + \nabla_Y X) + \Phi(X, Y) \tag{3.1}$$
$$\overline{g}(X^c, \alpha^v) = a\alpha(X), \ \overline{g}(\alpha^v, \beta^v) = 0, \tag{3.2}$$

for any vector fields X, Y and any differential 1-forms α, β on M, where $a \in \mathbb{R}^*$ and the real function $\Phi(X, Y)$ on T^*M is symmetric in X and Y. We assume $a > 0$.

Remark that the deformed Riemann extension generalizes both the Riemann extension introduced by Patterson, Walker in [8] (when $\Phi = 0$) and also the natural Riemann extension (see [7] and [9]) when $\Phi(X, Y) = bw(X)w(Y)$, $\forall X, Y \in \mathcal{X}(M)$, where $b \in \mathbb{R}$.

Proposition 3.1. *Let (M, ∇) be a manifold endowed with a symmetric linear connection. Then the total space of its cotangent bundle carries a para-Hermitian structure $(G_a, \bar{P})$ where G_a is the deformed Riemann extension with $\Phi_a(X, Y) = a(\nabla_X Y + \nabla_Y X)^v$ and $\bar{P}$ is defined by:*

$$\bar{P}X^c = X^c, \bar{P}\alpha^v = -\alpha^v, \ \forall X \in \mathcal{X}(M), \alpha \in \Omega^1(M). \tag{3.3}$$

Proof. One can see that $\bar{P}^2 = I$ and $\bar{P} \neq \pm I$ where I is the identity. We note that:

$$G_a(X^c, Y^c) = G_a(\alpha^v, \beta^v) = 0,\ G_a(X^c, \alpha^v) = a\alpha(X),$$
$$\forall X, Y \in \mathcal{X}(M), \alpha, \beta \in \Omega^1(M). \tag{3.4}$$

Moreover, $\bar{P}$ is skew-symmetric with respect to G_a:

$$G_a(\bar{P}U, \bar{P}V) = -G_a(U, V),\ \ \forall U, V \in \mathcal{X}(T^*M). \tag{3.5}$$

Hence $(G_a, \bar{P})$ is an almost para-Hermitian structure. Since the Nijenhuis tensor field of $\bar{P}$ vanishes identically it follows that $\bar{P}$ is integrable and therefore $(G_a, \bar{P})$ is a para-Hermitian structure which complete the proof. □

By using a similar but longer computation, we generalize Theorem 5.1 obtained in [2]:

Theorem 3.2. *If $X^v, Z^v \in \mathcal{F}(T^*M)$ are the vertical lifts of the vector field $X, Z \in \mathcal{X}(M)$ then:*

$$((grad\, Z^v)X^v)_{(x,w)} = \frac{1}{a}\{(\nabla_X Z + \nabla_Z X)^v - \frac{1}{a}\Phi(Z, X)\}_{(x,w)}, \tag{3.6}$$

where we used the characterization for $grad\, Z^v$ given by:

$$\overline{g}(grad\, Z^v, U) = UZ^v,\ \forall U \in \mathcal{X}(T^*M).$$

4 Harmonic maps and morphisms

We recall that a map $\varphi : (N, h) \to (\widetilde{N}, \widetilde{h})$ between (semi-)Riemannian manifolds is a harmonic map if the Euler-Lagrange operator $\tau(\varphi)$ defined as the trace (with respect to h) of the second fundamental form $\nabla d\varphi$ of φ vanishes identically, that is

$$\tau(\varphi) = \mathrm{trace}_h \nabla d\varphi = 0.$$

Definition 4.1. A map $\varphi : (N, h) \to (\widetilde{N}, \widetilde{h})$ between (semi-)Riemannian manifolds is:

(i) a harmonic morphism if for any harmonic function f defined (locally) on $\widetilde{N}$, its pull-back $f \circ \varphi$ is a (locally) harmonic function on N.

(ii) horizontally weakly conformal if there exists a function $\Lambda : N \to \mathbb{R}$ (called square dilatation) such that in any local coordinates $(y^{\widetilde{1}}, \ldots, y^{\widetilde{n}})$ on $\widetilde{N}$ one has:

$$h(\mathrm{grad}\varphi^\alpha, \mathrm{grad}\varphi^\beta) = \Lambda \widetilde{h}^{\alpha\beta},\ \alpha, \beta = \overline{1, \widetilde{n}},$$

see [1].

Theorem 4.2. *Let (M,g) be a Riemannian manifold and let $(T^*M,\overline{g})$ be the total space of its cotangent bundle endowed with the natural Riemannian extension constructed with the Levi-Civita connection ∇ of g. For any map $\varphi=(\varphi^1,\ldots,\varphi^k):(M,g)\to\mathbb{R}^k$, its associated map is defined at any point $(x,w)\in T^*M$ by:*

$$\widetilde{\varphi}(x,w)=((grad\,\varphi^1)_{(x,w)},\ldots,(grad\varphi^k)_{(x,w)}). \tag{4.1}$$

Then $\widetilde{\varphi}$ is a harmonic morphism if and only if $\widetilde{\varphi}$ is an eigenmap of the vertical lift of the Laplacian, i.e.

$$(\Delta\varphi)^v=\frac{b(n+1)}{2a}\widetilde{\varphi}, \tag{4.2}$$

and $(grad\,\varphi^1)^c,\ldots,(grad\,\varphi^k)^c$ are mutually orthogonal and of the name length.

Proof. Let $\overline{\nabla}$ be the Levi-Civita connection of $\overline{g}$. We take $\alpha\in\Omega^1(M)$ such that $\alpha_x=w$, (but the proof is independent of the choice of α that satisfies this condition). From the relation

$$(\mathrm{grad}\,Z^v)_{(x,w)}=\frac{1}{a}\{Z^c-2\overline{\nabla}_{\alpha^v}Z^c+\frac{b}{a}w(Z)\alpha^v\}_{(x,w)}$$

and (2.1) at any $(x,w)\in T^*M$, we obtain for any set of vector fields $\{Z_i\}_{i=\overline{1,k}}$:

$$\begin{aligned}\bar{g}\left(\mathrm{grad}\,Z_i^v,\mathrm{grad}\,Z_j^v\right)_{(x,w)}&=\frac{1}{a^2}\bar{g}(Z_i^c-2\bar{\nabla}_{\alpha^v}Z_i^c+cZ_i^v\alpha^v,Z_j^c-2\bar{\nabla}_{\alpha^v}Z_j^c\\&\quad+cZ_j^v\alpha^v)_{(x,w)}\\&=\frac{1}{a^2}\{\bar{g}\left(Z_i^c,Z_j^c\right)-2\bar{g}\left(Z_i^c,\bar{\nabla}_{\alpha^v}Z_j^c\right)\\&\quad+cZ_j^v\bar{g}\left(Z_i^c,\alpha^v\right)-2\bar{g}\left(\bar{\nabla}_{\alpha^v}Z_i^c,Z_j^c\right)\\&\quad+4\bar{g}\left(\bar{\nabla}_{\alpha^v}Z_i^c,\bar{\nabla}_{\alpha^v}Z_j^c\right)-2cZ_j^v\bar{g}\left(\bar{\nabla}_{\alpha^v}Z_i^c,\alpha^v\right)\\&\quad+cZ_i^v\bar{g}\left(\alpha^v,Z_j^c\right)-2cZ_i^v\bar{g}\left(\alpha^v,\bar{\nabla}_{\alpha^v}Z_j^c\right)\\&\quad+c^2Z_i^vZ_j^v\bar{g}\left(\alpha^v,\alpha^v\right)\}_{(x,w)}\end{aligned} \tag{4.3}$$

Using local coordinates, we can easily check:

$$\bar{g}(\mathbf{W},\beta^v)=\bar{g}(\mathbf{W},\mathbf{W})=0,\quad\forall\beta\in\Omega^1(M). \tag{4.4}$$

By using the definition of natural Riemann extension, the definition relation of $\overline{\nabla}$ and (4.4), we express some of the terms involved in (4.3):

$$\begin{aligned}\bar{g}\left(\bar{\nabla}_{\alpha^v}Z_i^c,\bar{\nabla}_{\alpha^v}Z_j^c\right)_{(x,w)}&=\bar{g}\left(\bar{\nabla}_{\alpha^v}Z_i^c,\alpha^v\right)_{(x,w)}=\bar{g}\left(\alpha^v,\bar{\nabla}_{\alpha^v}Z_j^c\right)_{(x,w)}\\&=\bar{g}\left(\alpha^v,\alpha^v\right)_{(x,w)}=0;\\\bar{g}\left(Z_i^c,Z_j^c\right)_{(x,w)}&=-aw\left(\nabla_{Z_i}Z_j+\nabla_{Z_j}Z_i\right)_x+bw\left(Z_i\right)_x w\left(Z_j\right)_x.\end{aligned}$$

Using the definition formula of $\overline{\nabla}$, we obtain

$$\bar{g}\left(Z_i^c, \bar{\nabla}_{\alpha^v} Z_j^c\right)_{(x,w)} = -a\alpha_x \left(\nabla_{Z_i} Z_j\right)_x + bw\left(Z_i\right)_x w\left(Z_j\right)_x$$

and $\bar{g}\left(Z_i^c, \alpha^v\right)_{(x,w)} = a\alpha_x \left(Z_i\right)_x$ where $i = \overline{1,k}$.

By substituting the previous relations (and also the above relations in which i and j replace each other) into (4.3), we obtain:

$$\begin{aligned}
&\bar{g}\left(\operatorname{grad} Z_i^v, \operatorname{grad} Z_j^v\right)_{(x,w)} \\
&= \frac{1}{a^2}\{-aw\left(\nabla_{Z_i} Z_j + \nabla_{Z_j} Z_i\right)_x + bw\left(Z_i\right)_x w\left(Z_j\right)_x \\
&+ 2aw\left(\nabla_{Z_i} Z_j\right)_x - 2bw\left(Z_i\right)_x w\left(Z_j\right)_x + 2aw\left(\nabla_{Z_j} Z_i\right)_x\} \\
&= \frac{1}{a^2}\left\{aw\left(\nabla_{Z_i} Z_j + \nabla_{Z_j} Z_i\right)_x - bw\left(Z_i\right)_x w\left(Z_j\right)_x\right\} \\
&= -\frac{1}{a^2}\bar{g}\left(Z_i^c, Z_j^c\right)_{(x,w)}, \quad i, j = \overline{1,k}.
\end{aligned} \tag{4.5}$$

Now, $Z_1^c, \ldots, Z_k^c$ are mutually orthogonal and of the same length on $(T^*M, \bar{g})$ if and only if there exists a real function $\Lambda : T^*M \to \mathbb{R}$ such that $\bar{g}\left(Z_i^c, Z_j^c\right) = \Lambda\delta_{ij}$, $i, j = \overline{1,k}$.

From (4.5), by taking $Z_i = \operatorname{grad}\varphi^i$, $i = \overline{1,k}$, we obtain that $\overline{\varphi}$ is horizontally weakly conformal if and only if $(\operatorname{grad}\varphi^1)^c, \ldots, (\operatorname{grad}\varphi^k)^c$ are mutually orthogonal and of the same length.

We recall from ([2], Corollary 4.2) that the vertical lift Y^v of a vector field $Y \in \mathcal{X}(M)$ is a harmonic function (with respect to a natural Riemann extension $\overline{g}$) on T^*M if and only if

$$(\operatorname{div} Y)^v = \frac{b(n+1)}{2a} Y^v.$$

Hence $\overline{\varphi}$ is a harmonic map with respect to $\overline{g}$ if and only if $\overline{\varphi}$ is an eigenmap of the vertical lift of the Laplacian

$$(\Delta\varphi)^v = ((\Delta\varphi^1)^v, \ldots, (\Delta\varphi^k)^v),$$

i.e. (4.2) is satisfied.

We complete the proof since any map between (semi-)Riemannian manifolds is a harmonic morphism if and only if it is harmonic and horizontally weakly conformal see [1]. □

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