Proceedings Book of International Workshop on Theory of Submanifolds (Volume: 1 (2016)) June 2–4, 2016, Istanbul, Turkey. Editors: Nurettin Cenk Turgay, Elif Özkara Canfes, Joeri Van der Veken and Cornelia-Livia Bejan
Recieved: January 31, 2017
Accepted: March 14, 2017
DOI: 10.24064/iwts2016.2017.4

On the Solutions to the $H_R = H_L$ Hypersurface Equation

Eva M. Alarcón, Alma L. Albujer, Magdalena Caballero

Eva M. Alarcón: Departamento de Matemáticas, Campus de Espinardo, Universidad de Murcia, 30100 Murcia, Spain, e-mail:evamaria.alarcon@um.es,
Alma L. Albujer: Departamento de Matemáticas, Campus Universitario de Rabanales, Universidad de Córdoba, 14071 Córdoba, Spain, e-mail:alma.albujer@uco.es,
Magdalena Caballero: Departamento de Matemáticas, Campus Universitario de Rabanales, Universidad de Córdoba, 14071 Córdoba, Spain, e-mail:magdalena.caballero@uco.es

Abstract. Spacelike hypersurfaces in the Lorentz-Minkowski (n+1)-dimensional space $\mathbb{L}^{n+1}$ can be endowed with another Riemannian metric, the one induced by the Euclidean space $\mathbb{R}^{n+1}$. The hypersurfaces with the same mean curvature with respect to both metrics can be locally determined by a smooth function u satisfying $|Du| < 1$, and being the solution to a certain partial differential equation. We call this equation the $H_R = H_L$ hypersurface equation. In the particular case in which $n = 2$ and both curvatures vanish, Kobayashi proved that the graphs determined by the solutions of such equation are open pieces of spacelike planes or helicoids, in the region where they are spacelike. In this manuscript we prove the existence of a family of solutions whose graphs have non-zero mean curvature, and we present an inequality relating the mean curvature to the width of the domain of certain solutions, those without critical points.
Keywords. Mean curvature · spacelike hypersurfaces · rotational hypersurfaces · elliptic partial differential equations.
MSC 2010 Classification. Primary: 53C42; Secondary: 35J93 · 53C50.

1 Introduction and background

Let us consider the differential operator given by

$$Q(u) = \operatorname{div}\left(\left(\frac{1}{\sqrt{1-|Du|^2}} - \frac{1}{\sqrt{1+|Du|^2}}\right) Du\right),$$

where $u \in \mathcal{C}^2(\mathbb{R}^n)$, and D, div and $|\cdot|$ stand for the gradient, the divergence and the Euclidean norm on $\mathbb{R}^n$, respectively. We are interested in studying the equation

$$Q(u) = 0, \text{ with } |Du| < 1. \tag{1.1}$$

The above divergence-type partial differential equation is not an arbitrary one, it has a geometrical meaning.

A hypersurface in the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ is said to be spacelike if its induced metric is a Riemannian one. Therefore, spacelike hypersurfaces in $\mathbb{L}^{n+1}$ can be endowed with two different Riemannian metrics, the metric induced by the Euclidean space $\mathbb{R}^{n+1}$ and the metric inherited from $\mathbb{L}^{n+1}$. Consequently, we can consider two different mean curvature functions on a spacelike hypersurface related to both metrics, H_R and H_L respectively.

On the other hand, it is well known that any spacelike hypersurface Σ in $\mathbb{L}^{n+1}$ can be locally described as a spacelike graph over an open subset of a spacelike hyperplane, which without loss of generality can be supposed to be the hyperplane $x_{n+1} = 0$ (see [4, Proposition 3.3]). Let u be the function that describes such a graph, then the spatiality condition becomes $|Du| < 1$. The functions H_R and H_L can be written in terms of the function u and its partial derivatives obtaining the expressions

$$H_R = \frac{1}{n}\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) \qquad \text{and} \qquad H_L = \frac{1}{n}\operatorname{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right). \tag{1.2}$$

Therefore, a spacelike graph determined by u satisfies $H_R = H_L$ if and only if u is a solution of (1.1). For this reason (1.1) is called the $H_R = H_L$ *hypersurface equation*. This equation is a quasilinear elliptic partial differential equation, everywhere except at those points at which Du vanishes, where the equation is parabolic, see [1].

As a particular case, we can consider the situation where the graph is simultaneously minimal and maximal, that is $H_R = H_L = 0$. The geometry of minimal and maximal graphs has been widely studied. One of the main results on minimal graphs is the well-known Bernstein theorem, proved by Bernstein [5] in 1915, which states that the only entire minimal graphs in $\mathbb{R}^3$ are the planes. Some decades later, in 1970, Calabi [7] proved its analogous version for spacelike graphs in the Lorentz-Minkowski space, the Calabi-Bernstein theorem, which states that the only entire maximal graphs in $\mathbb{L}^3$ are the spacelike planes. An important difference between both results is that the Bernstein theorem can be extended to minimal graphs in $\mathbb{R}^{n+1}$ up to dimension $n = 7$, as it was proved by Bombieri, di Giorgi and Giusti [6], but it is no longer true for higher dimensions. However, the Calabi-Bernstein theorem holds true for any dimension as it was proved by Calabi [7] for dimension $n \leq 4$, and by Cheng and Yau [8] for arbitrary dimension.

As an immediate consequence of the above results, we conclude that the only entire graphs that are simultaneously minimal in $\mathbb{R}^{n+1}$ and maximal in $\mathbb{L}^{n+1}$ are the spacelike hyperplanes.

Going a step further, we can consider spacelike graphs with the same constant mean curvature functions H_R and H_L. Heinz [11], Chern [9] and Flanders [10] proved that the only entire graphs with constant mean curvature H_R in $\mathbb{R}^{n+1}$ are the minimal graphs. The Lorentzian version of this fact is not true, since there are examples of entire spacelike graphs with constant mean curvature H_L in $\mathbb{L}^{n+1}$ which are not maximal, for instance the hyperbolic spaces. However, taking into account the Calabi-Bernstein theorem, we conclude again that the only complete spacelike hypersurfaces in $\mathbb{L}^{n+1}$ with the same constant mean curvature functions H_R and H_L are the spacelike hyperplanes.

Kobayashi [12] studied the same problem without assuming any global hypothesis. He showed that the graphs of the solutions to (1.1) with $H_R = H_L = 0$ are open pieces of a spacelike plane or of a helicoid, in the region where the helicoid is a spacelike surface. Recently, Albujer, Caballero and Sánchez [2, 3] have continued with the study of spacelike surfaces with the same mean curvature in $\mathbb{R}^3$ and in $\mathbb{L}^3$, not necessarily constant. On one hand, they have shown that the Gaussian curvature in $\mathbb{R}^3$ of those surfaces is always non-positive and have obtained several interesting consequences about the geometry of such surfaces. On the other hand, they have obtained results on the solutions to the $H_R = H_L$ surface equation, which are not derived from the sign of the Gaussian curvature.

In general dimension, Lee and Lee [13] have recently presented non-planar examples of simultaneously minimal and maximal spacelike graphs in the Lorentz-Minkowski space. Their examples can be seen as generalized ruled hypersurfaces, in fact they are a natural generalization of helicoids. However, there is no known classification of such hypersurfaces similar to Kobayashi's result. In [1] the authors have shown that those hypersurfaces do not have elliptic points and have obtained several interesting consequences about the geometry of such hypersurfaces, generalizing some results in [2].

In this manuscript we prove the existence of a solution to the $H_R = H_L$ hypersurface equation which constitutes the first evidence of the existence of examples with non-zero mean curvature, following the ideas of the example obtained in [3] in dimension 2. Finally, we generalize some results on the graphs of the solutions which are not a consequence of the non-existence of elliptic points, specifically Lemma 7, Theorem 8 and Corollary 1 from [2].

2 Preliminaries

Let $\mathbb{L}^{n+1}$ be the $(n+1)$-dimensional Lorentz-Minkowski space, that is, $\mathbb{R}^{n+1}$ endowed with the metric

$$\langle \cdot, \cdot \rangle_L = dx_1^2 + ... + dx_n^2 - dx_{n+1}^2,$$

where $(x_1, ..., x_{n+1})$ are the canonical coordinates in $\mathbb{R}^{n+1}$, and let $|\cdot|_L$ denote its norm. It is easy to see that the Levi-Civita connections of the Euclidean space $\mathbb{R}^{n+1}$ and the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ coincide, so we will just denote it by $\overline{\nabla}$.

A (connected) hypersurface Σ^n in $\mathbb{L}^{n+1}$ is said to be a spacelike hypersurface if $\mathbb{L}^{n+1}$ induces a Riemannian metric on Σ, which is also denoted by $\langle \cdot, \cdot \rangle_L$. Given a spacelike hypersurface Σ, we can choose a unique future-directed unit normal vector field N_L on Σ. The mean curvature function of Σ with respect to N_L is defined by

$$H_L = -\frac{1}{n}(k_1^L + ... + k_n^L),$$

where k_i^L, $i = 1, ..., n$, stand for the principal curvatures of $(\Sigma, \langle\cdot,\cdot\rangle_L)$.

The same topological hypersurface can also be considered as a hypersurface of the Euclidean space, that is $\mathbb{R}^{n+1}$ with its usual Euclidean metric. For simplicity, we will just denote the Euclidean space by $\mathbb{R}^{n+1}$, the Euclidean metric and the induced metric on Σ by $\langle\cdot,\cdot\rangle_R$, and its norm by $|\cdot|_R$. In such a case, Σ admits a unique upwards directed unit normal vector field, N_R. The mean curvature function of Σ with respect to N_R is defined by

$$H_R = \frac{1}{n}(k_1^R + ... + k_n^R),$$

where k_i^R, $i = 1, ..., n$, stand for the principal curvatures of $(\Sigma, \langle\cdot,\cdot\rangle_R)$.

It is interesting to observe that the mean curvature functions have an expression in terms of the normal curvatures of any set of orthogonal directions. Specifically,

$$H_L = -\frac{1}{n}(\kappa_{w_1}^L + \ldots + \kappa_{w_n}^L) \qquad \text{and} \qquad H_R = \frac{1}{n}(\kappa_{v_1}^R + \ldots + \kappa_{v_n}^R), \tag{2.1}$$

where $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ are orthonormal basis of $T_p\Sigma$ with respect to $\langle\cdot,\cdot\rangle_R$ and $\langle\cdot,\cdot\rangle_L$, respectively.

If our spacelike hypersurface is the graph of a smooth function $u \in \mathcal{C}^\infty(\Omega)$,

$$\Sigma_u = \{(x_1, ..., x_n, u(x_1, ..., x_n)) : (x_1, ..., x_n) \in \Omega\},$$

Ω being an open subset of the hyperplane $x_{n+1} = 0$, which can be identified with $\mathbb{R}^n$, it is easy to check that the spatiality condition is written as $|Du| < 1$, where D and $|\cdot|$ stand for the gradient operator and the norm in the Euclidean space $\mathbb{R}^n$, respectively. In this case, it is possible to get expressions for the normal vector fields N_L and N_R, as well as for the mean curvature functions H_L and H_R, in terms of u. Specifically, with a straightforward computation we get

$$N_L = \frac{(Du, 1)}{\sqrt{1 - |Du|^2}} \qquad \text{and} \qquad N_R = \frac{(-Du, 1)}{\sqrt{1 + |Du|^2}}. \tag{2.2}$$

And for the mean curvature functions we have

$$H_L = \frac{1}{n}\operatorname{div}\left(\frac{Du}{\sqrt{1 - |Du|^2}}\right) \qquad \text{and} \qquad H_R = \frac{1}{n}\operatorname{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right), \tag{2.3}$$

where div denotes the divergence operator in $\mathbb{R}^n$.

Let us observe that

$$\cosh\psi = \frac{1}{\sqrt{1 - |Du|^2}} \qquad \text{and} \qquad \cos\theta = \frac{1}{\sqrt{1 + |Du|^2}},$$

where ψ and θ denote the hyperbolic angle between N_L and $e_{n+1} = (0, ..., 0, 1)$ and the angle between N_R and e_{n+1}, respectively.

The following result can be found in [2], and will be used in Section 4.

Lemma 2.1. *[2, Lemma 2] Let Σ be a spacelike hypersurface in $\mathbb{L}^{n+1}$. Given $p \in \Sigma$ and $v \in T_p\Sigma$, let $\kappa_v^L(p)$ and $\kappa_v^R(p)$ denote the normal curvatures at p in the direction of v with respect to $\langle\cdot,\cdot\rangle_L$ and $\langle\cdot,\cdot\rangle_R$, respectively. Then*

$$\frac{|v|_R^2}{\cos\theta(p)}\,\kappa_v^R(p) = -\frac{|v|_L^2}{\cosh\psi(p)}\,\kappa_v^L(p).$$

3 A solution with non-zero mean curvature

Let us consider rotationally invariant spacelike graphs with respect to a vertical axis. Therefore, we can assume without loss of generality that the graph Σ_u^* is determined by a function

$$u(x_1, ..., x_n) = f(r), \quad r = x_1^2 + ... + x_n^2, \tag{3.1}$$

being $f \in \mathcal{C}^\infty(I)$ for certain $I \subseteq [0, +\infty)$. In this case, $|Du| < 1$ reads $4\left(f'(r)\right)^2 r < 1$ and the $H_R = H_L$ hypersurface equation yields

$$\frac{2f''r + f'n + 4(f')^3 r(n-1)}{(1 + 4(f')^2 r)^{3/2}} = \frac{2f''r + f'n - 4(f')^3 r(n-1)}{(1 - 4(f')^2 r)^{3/2}}. \tag{3.2}$$

It can be checked that, given any set of initial conditions $(r_0, f(r_0), f'(r_0))$ such that $r_0 > 0$, $f'(r_0) \neq 0$ and $4\left(f'(r_0)\right)^2 r_0 < 1$, there exists a local solution of (3.2) by the Picard-Lindelöf theorem.

It is interesting to observe that these examples cannot be entire because of the following theorem which can be found in [1].

Theorem 3.1. *The only entire spacelike graphs Σ_u determined by a function u given by* (3.1) *such that $H_R = H_L$ are the horizontal hyperplanes.*

4 On the width of the domain of the solutions

We define the *width* of a set in $\mathbb{R}^n$ as the supremum of the diameter of the closed balls contained in it. This is an intuitive definition which is a generalization of the classical concept of width for a convex body, see [15].

Let u be a solution to (1.1) over an open set $\Omega \subseteq \mathbb{R}^n$, Σ_u its graph and $\pi : \Sigma_u \longrightarrow \Omega$ the canonical projection. We define Σ_u^* as the graph of u over the following open set

$$\Omega^* = \{(x_1, \dots, x_n) \in \Omega : Du(x_1, \dots, x_n) \neq 0\}. \tag{4.1}$$

The goal of this section is to give an upper bound for the width of the set Ω^*. Before stating our main result, we get some previous local computations involving the Riemannian and Lorentzian normal curvatures of Σ_u^* in some privileged directions. As well as a lemma relating the mean curvature of Σ_u to that of its level hypersurfaces.

Given $p \in \Sigma_u^*$, we consider its corresponding level hypersurface contained in $\mathbb{R}^n$, $\widetilde{S_c}$, and its lifting to Σ_u, S_c. We will work in a neighborhood of p, hence we can assume that S_c lies on Σ_u. Since $Du \neq 0$ in Ω^*, this distribution is integrable, so we can consider the integral curve through $\pi(p)$. We denote by α its lifting to Σ_u^*. Notice that $\alpha' = (Du, |Du|^2) \circ \pi$.

Therefore, we have two submanifolds of Σ_u^*, namely S_c and α, defined on a neighborhood of p which are orthogonal at p for both $\langle\cdot,\cdot\rangle_R$ and $\langle\cdot,\cdot\rangle_L$. Now, let $\{e_1, \dots, e_{n-1}\}$ be an orthonormal basis of $T_{\pi(p)}\widetilde{S_c}$. The vectors $\{(e_1, 0), \dots, (e_{n-1}, 0)\}$ constitute an orthonormal basis of T_pS_c in both $\mathbb{R}^{n+1}$ and $\mathbb{L}^{n+1}$, and are orthogonal to α' for both metrics. Then, Lemma 2.1 gives us the following relationships, where we have omitted the point p on behalf of simplicity

$$\kappa^R_{(e_i,0)} = -\frac{\cos\theta}{\cosh\psi}\,\kappa^L_{(e_i,0)} = -\sqrt{\frac{1-|Du|^2}{1+|Du|^2}}\,\kappa^L_{(e_i,0)}, \qquad i = 1, \dots, n-1 \qquad \text{and}$$

$$\kappa^R_{\alpha'} = -\frac{|\alpha'|^2_L}{|\alpha'|^2_R}\frac{\cos\theta}{\cosh\psi}\,\kappa^L_{\alpha'} = -\left(\frac{1-|Du|^2}{1+|Du|^2}\right)^{\frac{3}{2}}\,\kappa^L_{\alpha'}.$$

By denoting $A = \sqrt{\frac{1-|Du|^2}{1+|Du|^2}}$, we rewrite the previous expressions as

$$\kappa^R_{(e_i,0)} = -A\,\kappa^L_{(e_i,0)}, \qquad i = 1, \dots, n-1 \qquad \text{and} \qquad \kappa^R_{\alpha'} = -A^3\,\kappa^L_{\alpha'}. \tag{4.2}$$

As we are dealing with orthogonal directions at p for both $\langle\cdot,\cdot\rangle_R$ and $\langle\cdot,\cdot\rangle_L$, and u is a solution of the $H_R = H_L$ hypersurface equation, from (2.1) we get

$$-\kappa^L_{(e_1,0)} - \dots - \kappa^L_{(e_{n-1},0)} - \kappa^L_{\alpha'} = \kappa^R_{(e_1,0)} + \dots + \kappa^R_{(e_{n-1},0)} + \kappa^R_{\alpha'},$$

which jointly with (4.2) implies

$$-\kappa^L_{\alpha'} = \frac{1}{A^2+A+1}(\kappa^L_{(e_1,0)} + \dots + \kappa^L_{(e_{n-1},0)}). \tag{4.3}$$

Lemma 4.1. *Let Σ_u be a spacelike graph in $\mathbb{L}^{n+1}$ over a domain $\Omega \subseteq \mathbb{R}^n$ such that $H_R = H_L$. If $\widetilde{S_c}$ denotes the level hypersurface $u \equiv c$ in Ω^* and H_c is its mean curvature, then*

$$|H_L| \leq \frac{n-1}{n\sqrt{2}}|H_c| \circ \pi \tag{4.4}$$

and the equality is hold if and only if $H_L = 0$.

Proof. We work at a point $p \in S_c$ and we follow the notation introduced at the beginning of this section. For each $i = 1, \dots, n$ we take a curve in $\widetilde{S_c}$, $\widetilde{\alpha}_i$, with $\widetilde{\alpha}_i(0) = p$ and $\widetilde{\alpha}_i{}'(0) = e_i$. Let α_i be its lifting to S_c. Notice that $\alpha_i' = (\widetilde{\alpha}_i', 0)$.

It is possible to relate the Lorentzian normal curvature $\kappa^L_{(e_i,0)}$ of Σ_u at p in the direction of α_i' with the normal curvature $\kappa^c_{e_i}$ of $\widetilde{S_c}$ at $\pi(p)$ in the direction of $\widetilde{\alpha_i}'$:

$$\kappa^L_{(e_i,0)} = \langle \overline{\nabla}_{t_i} t_i, N_L\rangle_L = \frac{|Du|}{\sqrt{1-|Du|^2}} \left\langle D_{\widetilde{t_i}}\, \widetilde{t_i}, \frac{Du}{|Du|} \right\rangle_{\mathbb{R}^n} \circ\pi = \frac{|Du|}{\sqrt{1-|Du|^2}} \kappa^c_{e_i} \circ\pi.$$

Here D and $\langle \cdot,\cdot\rangle_{\mathbb{R}^n}$ stand for the Levi-Civita connection and the usual metric of the Euclidean space $\mathbb{R}^n$, respectively, $t_i = \dfrac{\alpha_i'}{|\alpha_i'|_L}$, $\widetilde{t_i} = \dfrac{\widetilde{\alpha_i}'}{|\widetilde{\alpha_i}'|}$ and $\dfrac{Du}{|Du|}$ is the unitary normal vector field to $\widetilde{S_c}$ in $\mathbb{R}^n$.

Therefore, from (4.3) we get

$$n|H_L| = (n-1)\frac{A+1}{A^2+A+1}\frac{|Du|}{\sqrt{1+|Du|^2}}|H_c|\circ\pi \leq (n-1)f(|Du|)\,|H_c|\circ\pi,$$

where $f(x) = \frac{x}{\sqrt{1+x^2}}$. Since f is increasing and $|Du|<1$, we get (4.4).

□

Theorem 4.2. *Let u be a solution to the $H_R = H_L$ hypersurface equation defined on an open set $\Omega \subseteq \mathbb{R}^n$. Then*

$$\mathrm{width}(\Omega^*) \leq \frac{\sqrt{2}\,(n-1)}{n\,\inf_{\Omega^*}|H_L|}. \tag{4.5}$$

Proof. If $\inf_{\Omega^*}|H_L| = 0$, there is nothing to prove.

Otherwise, we have $|H_L| \geq \inf_{\Omega^*}|H_L| = C > 0$ in Σ_u^*. And, as a consequence of (4.4), we get

$$|H_c| > \frac{nC\sqrt{2}}{n-1} > 0 \quad \text{in} \quad \Omega^*. \tag{4.6}$$

First of all, let us notice that Ω^* is an open set of $\mathbb{R}^n$. We consider all the level hypersurfaces in Ω^*, we order them by the value of u on each of them and we orient them in a way such that its normal vectors point to the direction on which u decreases.

We proceed by reductio ad absurdum assuming that the width of Ω^* is bigger than $\dfrac{(n-1)\sqrt{2}}{nC}$. Then, there exists a point $q \in \Omega^*$ such that $\bar{B}_q = \bar{B}_q\left((n-1)/n\sqrt{2}\,C\right) \subset \Omega^*$. Since $\bar{B}_q$ is compact, u attains a maximum in it. Even more, Du does not vanish in $B_q = B_q\left((n-1)/n\sqrt{2}\,C\right)$, and so this extremal value is only attained on the boundary of the ball.

We pick a point p at which a maximum is attained. The level hypersurface through p lies in $\Omega^* \setminus B_q$. And so, it is tangent to the boundary of the ball at p. The normal vector to the hypersurface at p points to the interior of the ball, see Figure 1. Consequently, using the tangency principle (see [14, Theorem 3.2.4]

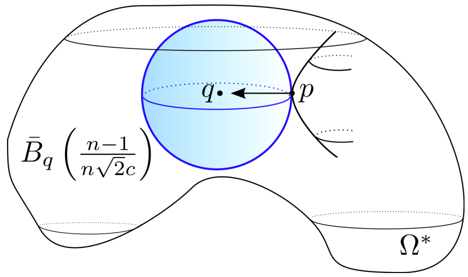

Figure 1: Level hypersurface at p.

for the 2-dimensional case), inequality (4.6) implies that $H_c \leq -n\sqrt{2}C/(n-1)$ at p. Analogously, we get that $H_c \geq n\sqrt{2}C/(n-1)$ at $\bar{p}$, $\bar{p}$ being a point at which u attains a minimum in the ball. By a continuity argument, there is a point in the ball at which H_c vanishes, which is a contradiction.

□

As a direct consequence of Theorem 4.2, we get the following results.

Corollary 4.3. *Let u be a solution to the $H_R = H_L$ hypersurface equation defined on an open set $\Omega \subseteq \mathbb{R}^n$ and assume that Ω^* is a set of infinite width. Then* $\inf_{\Sigma_u} |H_L| = 0$.

Equivalently, there do not exist spacelike graphs satisfying $H_R = H_L$, $|H_L| \geq C$ for a certain constant $C > 0$ and $\mathrm{width}(\Omega^*) = \infty$.

Corollary 4.4. *Let u be a solution to the $H_R = H_L$ hypersurface equation defined on an open set $\Omega \subseteq \mathbb{R}^n$ with constant mean curvature. Then*

$$\mathrm{width}(\Omega^*) \leq \frac{\sqrt{2}\,(n-1)}{n\,|H_L|}.$$

5 Acknowledgments

The authors would like to thank the anonymous reviewers for their valuable comments and suggestions to improve the quality of the paper.

The first author has been supported by a PhD grant of the University of Murcia, Ayuda de Iniciación a la Investigación. The second author is partially supported by MINECO/FEDER project reference MTM2015-65430-P, Spain, and Fundación Séneca project reference 19901/GERM/15, Spain. Her work is a result of the activity developed within the framework of the Program in Support of Excellence Groups of the Región de Murcia, Spain, by Fundación Séneca, Science and Technology Agency of the Región de Murcia. The first and the third authors are partially supported by the Spanish Ministry of Economy and Competitiveness and European Regional Development Fund (ERDF), project MTM2013-47828-C2-1-P.

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